Self-Similar Gaussian Random Fields and Their Properties

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International Conference on Self-Similar Processes and Its Applications

Angers, France, July 20–24, 2009

Outline

1. Properties of self-similarity

2. Dependence structures of (time anisotropic) Gaussian fields

3. Properties of strong local nondeterminism

4. Uniform Hausdorff dimension results

5. Exact Hausdorff measure functions

1. Self-similarities of Random Fields

1.1 Definition of self-similarity

Let $H > 0$ be a constant. A random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ is called H-self-similar if for every constant $c > 0$,

$$\left\{X(ct),\, t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c^H X(t),\, t \in \mathbb{R}^N\right\},$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions.

Example 1. Fractional Brownian motion $X^H = \{X^H(t),\, t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$, which is a centered Gaussian field with covariance function

$$\mathbb{E}\Big[X_i^H(s) X_j^H(t)\Big] = \frac{1}{2}\,\delta_{ij}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, is H-self-similar.

1.2 Operator-self-similarity in space

An (N, d)-random field $\{X(t), t \in \mathbb{R}^N\}$ is called *operator self-similar in the space-variable* if there exists a $d \times d$ matrix $D = (d_{ij})$ such that for all constants $c > 0$,

$$\Big\{X(c\,t), t \in \mathbb{R}^N\Big\} \stackrel{d}{=} \Big\{c^D\,X(t), t \in \mathbb{R}^N\Big\}.$$

In the above, c^D is the linear operator defined by

$$c^D = \sum_{n=0}^{\infty} \frac{(\ln c)^n D^n}{n!}.$$

The linear operator D is called a space-variable self-similarity exponent [which may not be unique].

Example 2. Gaussian fields with fBm components

Let $X_1, \ldots, X_d$ be independent N-parameter fractional Brownian motions in $\mathbb{R}$ with indices $\alpha_1, \ldots, \alpha_d$, respectively.

We define an (N, d)-Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ by

$$X(t) = \Big(X_1(t), \ldots, X_d(t)\Big), \qquad t \in \mathbb{R}^N.$$

Then X is operator-self-similar with $D=$ the diagonal matrix with entries $\alpha_1, \ldots, \alpha_d$ on the diagonal.

When $\alpha_1, \ldots, \alpha_d$ are not the same, X is anisotropic in the space variable.

General operator-fractional Brownian motions were constructed in Mason and Xiao (2001), Bahadoran, Benassi and Dębicki (2003), Didier and Pipiras (2007a, b).

1.3 Operator-self-similarity in time

An (N, d)-random field $\{X(t), t \in \mathbb{R}^N\}$ is called *operator self-similar in the time-variable* if there exists an $N \times N$ matrix E such that for all constants $c > 0$,

$$\Big\{X(c^E t),\, t \in \mathbb{R}^N\Big\} \overset{d}{=} \Big\{c\, X(t),\, t \in \mathbb{R}^N\Big\}.$$

The linear operator E is called a time-variable self-similarity exponent [which may not be unique].

Examples: Fractional Brownian sheets;

Solution to stochastic heat equation;

Operator-scaling fields: Biermé, Meerschaert and Scheffler (2007).

1.3.1 Fractional Brownian sheets

$W^H = \left\{W^H(t), t \in \mathbb{R}^N\right\}$ is a mean 0 Gaussian random field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^H(s)W^H(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right),$$

where $H = (H_1, \ldots, H_N) \in (0,1)^N$.

W^H has the *operator-scaling property*: For all constants $c > 0$,

$$\left\{W^H(c^E t), t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c\,W^H(t),\ t \in \mathbb{R}^N\right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$.

1.3.2 Gaussian fields with stationary increments

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. If $R(s,t) = \mathbb{E}\Big[X(s)X(t)\Big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \int_{\mathbb{R}^N} (e^{is\cdot\lambda} - 1)(e^{-it\cdot\lambda} - 1)\Delta(d\lambda) + \langle s, Qt\rangle,$$

where Q is an $N \times N$ non-negative definite matrix and $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1 + |\lambda|^2}\, \Delta(d\lambda) < \infty. \tag{1}$$

The measure Δ is called the *spectral measure* of X.

It follows that X has the stochastic integral representation:

$$\left\{X(t), t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{\int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) \mathcal{M}(d\lambda) + \langle \mathbf{Y}, t\rangle,\ t \in \mathbb{R}^N\right\},$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\mathcal{M}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure and $\mathbf{Y}$ is an N-dimensional Gaussian random vector with mean 0 and covariance matrix Q, which is independent of $\mathcal{M}$.

Gaussian fields with stationary increments can be constructed by choosing spectral measure Δ.

For example, if Δ has a density function

$$f_\alpha(\lambda) = c(\alpha, N)\, |\lambda|^{-(2\alpha+N)},$$

where $\alpha \in (0,1)$ and $c(\alpha, N) > 0$ is a normalizing constant, then X is fBm with index α.

Remarks

- One can also construct random fields with more general scaling properties [e.g. Li and Xiao, 2009].

- Properties of Gaussian fields with space or time anisotropy are different from those of isotropic Gaussian fields such as fBm.

  - space-anisotropic GRF: Cuzick (1978), Adler (1981), Xiao (1995), Mason and Xiao (2001), ...

  - time-anisotropic GRF: Ayache and Xiao (2005), Wu and Xiao (2007, 2009), Xiao (2009), Biermé, Lacaux and Xiao (2009), ...

2. Dependence structures of Gaussian fields with time anisotropy

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field in $\mathbb{R}^d$ defined by

$$X(t) = \Big(X_1(t), \ldots, X_d(t)\Big), \qquad t \in \mathbb{R}^N, \tag{2}$$

where $X_1, \ldots, X_d$ are independent copies of X_0. Denote

$$\sigma^2(s,t) = \mathbb{E}\Big(X_0(s) - X_0(t)\Big)^2.$$

Given constants $0 < H_1 \le \ldots \le H_N < 1$, define a metric ρ on $\mathbb{R}^N$:

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in \mathbb{R}^N. \tag{3}$$

Three basic conditions

(C1). $\exists$ positive constants $c_{2,1}$ and $c_{2,2}$ such that for all $s, t \in [\varepsilon, 1]^N$,

$$c_{2,1}\,\rho(s,t)^2 \le \sigma^2(s,t) \le c_{2,2}\,\rho(s,t)^2.$$

(C2). $\exists$ $c_{2,3} > 0$ such that for all $s, t \in [\varepsilon, 1]^N$,

$$\mathrm{Var}\Big(X_0(t)\Big|X_0(s)\Big) \ge c_{2,3}\,\rho(s,t)^2.$$

(C3). $\exists\ c_{2,4} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X_0(t^n)\Big) \geq c_{2,4} \sum_{j=1}^{N} \min_{0 \leq k \leq n} \Big|u_j - t_j^k\Big|^{2H_j},$$

where $t_j^0 = 0$ for every $j = 1, \ldots, N$.

or

(C3′). $\exists\ c_{2,5} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X_0(t^n)\Big) \geq c_{2,5} \min_{0 \leq k \leq n} \rho(u,\, t^k)^2,$$

where $t^0 = 0$ and ρ is the metric on $\mathbb{R}^N$ defined in (3).

Condition (C1) and/or (C2) are sufficient for determining

- (upper bound for) sharp modulus of continuity.

- hitting probability estimates (e.g., Given $F \subset \mathbb{R}^d$, when can we have $\mathbb{P}\Big\{X([\varepsilon, 1]^N) \cap F \neq \varnothing\Big\} > 0$?): Xiao (2009) and Biermé, Lacaux and Xiao (2009).

- Hausdorff dimension of the images, graph, level sets

- existence of local times

However, Condition (C3) or (C3$'$) will be needed for establishing stronger or more precise results such as

- exact modulus of continuity
- uniform dimension results
- exact Hausdorff measure functions
- sharp Hölder conditions for the local times, etc.

3. Properties of strong local nondeterminism

3.1 Fractional Brownian sheets (FBS)

Wu and Xiao (2007) proved that FBS satisfies the *sectorial local nondeterminism* [Condition (C3)].

Theorem 3.1 For $\varepsilon > 0$, $\exists\ c_{3,1} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$,

$$\mathrm{Var}\Big(W^H(u)\,\Big|\,W^H(t^1), \ldots, W^H(t^n)\Big) \geq c_{3,1}\ \sum_{j=1}^{N} \min_{0 \leq k \leq n} \left|u_j - t_j^k\right|^{2H_j},$$

where $t_j^0 = 0$ for every $j = 1, \ldots, N$.

3.2 Anisotropic Gaussian fields with stationary increments

Theorem 3.2 [Xiao (2009), Xue and Xiao (2009)] Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and spectral density $f(\lambda)$.

If $\exists\ (H_1, \cdots, H_N) \in (0,1]^N$ such that

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{2+Q}}, \qquad \lambda \in \mathbb{R}^N \text{ with } |\lambda| \geq 1, \tag{4}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, then $\exists\ c_{3,2} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c_{3,2} \min_{0 \leq k \leq n} \rho(u, t^k)^2, \tag{5}$$

where $t^0 = 0$.

Proof of Theorem 3.2 Denote $r \equiv \min_{0 \le k \le n} \rho(u, t^k)$. It is sufficient to prove that for all $a_k \in \mathbb{R}$ $(1 \le k \le n)$,

$$\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k\, X(t^k)\Big)^2 \ge c_{3,2}\, r^2. \tag{6}$$

By the stochastic integral representation of X, the left hand side of (6), up to a constant, can be written as

$$\begin{aligned} &\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \\ &= \int_{\mathbb{R}^N} \Big| e^{i\langle u,\lambda\rangle} - 1 - \sum_{k=1}^{n} a_k\, (e^{i\langle t^k,\, \lambda\rangle} - 1)\Big|^2 f(\lambda)\, d\lambda. \end{aligned} \tag{7}$$

Hence, we only need to show

$$\int_{\mathbb{R}^N} \left| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k \, e^{i\langle t^k, \lambda\rangle} \right|^2 f(\lambda)\, d\lambda \geq c_{3,2}\, r^2, \tag{8}$$

where $t^0 = 0$ and $a_0 = -1 + \sum_{k=1}^n a_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0,1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0,1)$.

Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ and decays rapidly as $|\lambda| \to \infty$.

Let A be the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q}\delta(r^{-A}t)$. By the inverse Fourier transform,

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r^A\lambda)\, d\lambda.$$

Since $\min\{\rho(u, t^k) : 0 \le k \le n\} = r$, we have

$$\delta_r(u - t^k) = 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Hence,

$$\begin{aligned} I &= \int_{\mathbb{R}^N} \left(e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle} \right) e^{-i\langle u,\lambda\rangle}\, \widehat{\delta}(r^E\lambda)\, d\lambda \\ &= (2\pi)^N \left(\delta_r(0) - \sum_{k=0}^{n} a_k\, \delta_r(u - t^k) \right) \qquad (9) \\ &= (2\pi)^N\, r^{-Q}. \end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
I^2 &\le \int_{\mathbb{R}^N} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle}\Big|^2 f(\lambda)\, d\lambda \cdot \int_{\mathbb{R}^N} \frac{1}{f(\lambda)} \Big|\widehat{\delta}(r^A\lambda)\Big|^2 d\lambda \\
&\le \mathbb{E}\bigg(X(u) - \sum_{k=1}^{n} a_k X(t^k)\bigg)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{1}{f(r^{-A}\lambda)} \Big|\widehat{\delta}(\lambda)\Big|^2 d\lambda \\
&\le c\, \mathbb{E}\bigg(X(u) - \sum_{k=1}^{n} a_k X(t^k)\bigg)^2 \cdot r^{-2Q-2}.
\end{aligned}$$

We square both sides of (9) and use the above to obtain

$$(2\pi)^{2N}\, r^{-2Q} \le c\, r^{-2Q-2}\, \mathbb{E}\bigg(X(u) - \sum_{k=1}^{n} a_k X(t^k)\bigg)^2.$$

This proves (8) and hence the theorem.

The rest of the talk is about applications of Conditions (C3) and/or (C3′).

4. Uniform Hausdorff dimension results

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by (2). In Wu and Xiao (2007) and Xiao (2009), it is proved that if (C1) holds, then for every Borel set $E \subseteq (0, \infty)^N$,

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \dim_{\mathrm{H}}^{\rho} E\right\} \quad \text{a.s.}, \tag{10}$$

where $\dim_{\mathrm{H}}^{\rho} E$ is the Hausdorff dimension of E under the metric ρ [see, e.g., Wu and Xiao (2007)].

Note that the exceptional null probability event in (10) depends on E.

Question: When can we have

$$\mathbb{P}\left\{\dim_{\mathrm{H}} X(E) = \min\left\{d, \dim_{\mathrm{H}}^{\rho} E\right\} \ \ \forall\, E \subseteq (0, \infty)^N\right\} = 1?$$

Theorem 4.1. If X satisfies Condition (C1)–(C3) and

$$\sum_{\ell=1}^{N} \frac{1}{H_\ell} \le d,$$

then with probability 1

$$\dim_{\mathsf{H}} X(E) = \dim_{\mathsf{H}}^{\rho} E \quad \text{for all Borel sets} \quad E \subseteq (0,\infty)^N.$$

Sketch of Proof Half of the proof is easy: By the uniform modulus of continuity of X,

$$\mathbb{P}\left\{\dim_{\mathsf{H}} X(E) \le \dim_{\mathsf{H}}^{\rho} E \ \ \forall\ E \subseteq (0,\infty)^N\right\} = 1.$$

In order to prove the lower bound, we will make use of the following lemma, whose proof depends on (C3).

Lemma 4.2. Let $\delta > 0$ and $0 < 1 - \delta < \eta < 1$. Then with probability 1, for all large enough n and all balls $O \subseteq \mathbb{R}^d$ of radius $2^{-n\eta}$, $X^{-1}(O)$ can only intersect at most $2^{n\delta d}$ rectangles I_k^n, where

$$I_k^n = \left\{ t \in [\varepsilon, 1]^N : \ \frac{k_\ell - 1}{2^{n/H_\ell}} \le t_\ell \le \frac{k_\ell}{2^{n/H_\ell}}, \quad \ell = 1, \ldots, N \right\}.$$

Key: For any n points $t^1, \ldots, t^n$ of the form

$$t^j = \left(2^{-\frac{n}{H_1}} k_1^j, \ldots, 2^{-\frac{n}{H_N}} k_N^j \right) \in [\varepsilon, 1]^N,$$

how to estimate

$$\mathbb{P}\left\{ \left| X(t^i) - X(t^j) \right| < 3 \cdot 2^{-n\eta}, \quad \forall i \ne j \le n \right\}?$$

5. Exact Hausdorff measure functions

Properties of SLND are useful for determining the exact Hausdorff measure functions for $X([0,1]^N)$, Gr$X([0,1]^N)$ and $X^{-1}(x)$ $(x \in \mathbb{R}^d)$.

For example, we know that if X is either FBS W^H or the Gaussian field in Theorem 3.2, then

$$\dim_{\mathrm{H}} X([0,1]^N) = \min\left\{d,\ \sum_{j=1}^{N} \frac{1}{H_j}\right\} \quad \text{a.s.}$$

Question: Is there a gauge function φ such that

$$0 < \varphi\text{-}m\Big(X([0,1]^N)\Big) < \infty \qquad \text{a.s.?}$$

Here φ is called an exact Hausdorff measure function for $X([0,1]^N)$.

Theorem 5.1 Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field in $\mathbb{R}^d$ defined by (2). We assume that X has stationary increments with spectral density $f(\lambda)$ which satisfies (4).

(i). If $\sum_{j=1}^N \frac{1}{H_j} > d$, then a.s. $X([0,1]^N)$ has positive Lebesgue measure (even has interior points).

(ii). If $\sum_{j=1}^N \frac{1}{H_j} < d$, then there exists positive and finite constants $c_{5,1}$ and $c_{5,2}$ such that

$$c_{5,1} \le \varphi\text{-}m\left(X\left([0,1]^N\right)\right) \le c_{5,2} \quad \text{a.s.},$$

where $\varphi(r) = r^{\sum_{j=1}^N \frac{1}{H_j}} \log\log 1/r$.

The proof of Theorem 5.1 is divided into proving the lower and upper bounds separately.

Both parts rely on the following property of strong local non-determinism (Theorem 3.2).

Lemma 5.2 Under the assumptions of Theorem 5.1, $\exists\, c_{5,3} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [\varepsilon, 1]^N$,

$$\mathrm{Var}\Big(X_1(u) \,\Big|\, X_1(t^1), \ldots, X_1(t^n)\Big) \geq c_{5,3} \min_{0 \leq k \leq n} \rho(u,\, t^k)^2,$$

where $t^0 = 0$ and $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$.

Lemma 5.3 Let the assumptions of Theorem 5.1 hold. For every $t_0 \in [0,1]^N$, define

$$T_{t_0}(r) = \int_{[0,1]^N} \mathbf{1}_{\{|X(t)-X(t_0)| \le r\}}\, dt,$$

the sojourn time of X in the ball $B(X(t_0), r)$. Then with probability 1,

$$\limsup_{r \to 0} \frac{T_{t_0}(r)}{\varphi(2r)} \le c_{5,4} < \infty.$$

The proof of Lemma 5.3 replies on proving

$$\mathbb{E}\left(T_{t_0}(r)^n\right) = \int_{[0,1]^{Nn}} \mathbb{P}\left\{|X(t^j) - X(t_0)| \le r,\ 1 \le j \le n\right\} dt^1 \cdots dt^n$$
$$\le c_{5,5}^n\, n!\, r^{Qn} \tag{11}$$

for all integers $n \ge 1$.

From Lemma 5.3, together with the upper density theorem of Rogers and Taylor (1961), we derive

$$\varphi\text{-}m\left(X\left([0,1]^N\right)\right) \geq c_{5,4}^{-1} \quad \text{a.s.}$$

The estimate of $\mathbb{E}\left(T_{t_0}(r)^n\right)$ for fractional Brownian sheet W^H is different from (11):

$$\mathbb{E}\left(T_{t_0}(r)^n\right) \leq c_{5,6}^n\,(n!)^N\,r^{Qn}.$$

This suggests that the exact Hausdorff measure function for $W^H([0,1]^N)$, in the transient case of $Q < d$, is

$$\psi(r) = r^{\sum_{j=1}^N \frac{1}{H_j}}\Big(\log\log 1/r\Big)^N.$$

Thank You!