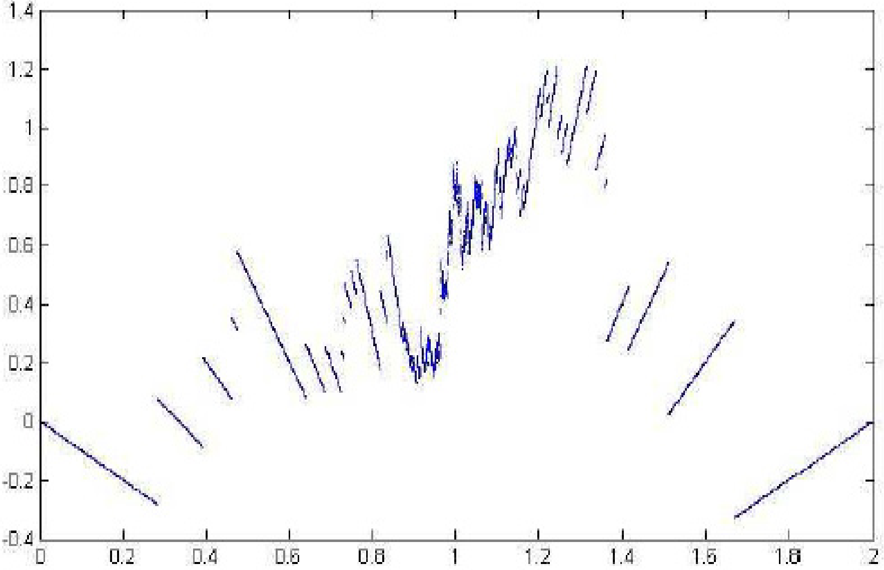

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# CONDITIONING OF QUADRATIC HARNESSES: BRIDGES AND GLUING CONSTRUCTIONS


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Self-similar processes
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joint with W. Bryc (Cincinnati, USA)

# Plan

# Plan

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1. Quadratic harness - basics
2. Time and space transformations
3. Bridges
4. Gluing
5. Bi-Pascal process

# Plan

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# Harness (standard)

An integrable stochastic process $X = (X_t)_{t \geq 0}$ is a harness if for any $s \leq t \leq u$

$$E(X_t | \mathcal{F}_{s,u}) = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u \ , \qquad (1)$$

where $\mathcal{F}_{s,u} = \sigma(X_\alpha, \ \alpha \notin (s,u))$ for $s \leq t \leq u$.

The process $X$ is a standard harness if additionally it is square-integrable and

$$E(X_t) = 0 \quad \text{and} \quad E(X_s X_t) = s \wedge t \qquad (2)$$

# Harness (standard)

Equivalently, for

$$\Delta_{s,u}(X) = \frac{X_u - X_s}{u - s} \quad \text{and} \quad \widetilde{\Delta}_{s,u}(X) = \frac{uX_s - sX_u}{u - s}$$

the harness condition can be written as

$$E(X_t | \mathcal{F}_{s,u}) = \langle \underline{t}, \underline{\Delta}_{s,u}(X) \rangle \ ,$$

where

$$\underline{t} = \begin{bmatrix} t \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{\Delta}_{s,u}(X) = \begin{bmatrix} \Delta_{s,u}(X) \\ \widetilde{\Delta}_{s,u}(X) \end{bmatrix}$$

# Quadratic harness (standard)

If additionally for $s \le t \le u$ there exist a quadratic function $Q_{t,s,u}$ such that

$$Var(X_t|\mathcal{F}_{s,u}) = Q_{t,s,u}(X_s, X_u) \tag{3}$$

we say that $X$ is (standard) quadratic harness.

Examples: Wiener process, Poisson type process, Lévy-Meixner processes, free Brownian motion, free Poisson process, $q$-Gaussian process, bi-Poisson processes, bi-Pascal process, Askey-Wilson type processes.

All are Markov, all are uniquely determined by moments.

Constructions: based on o-g polynomials for marginal distributions and for transition probabilities.

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# Quadratic harness (standard) - 5 parameters

There exist 5 numbers

$$\eta,\ \theta \in \mathbf{R}\ , \quad \sigma,\ \tau \geq 0\ , \quad \gamma \leq 1 + 2\sqrt{\sigma\tau}$$

such that

$$Var(X_t | \mathcal{F}_{s,u}) = F_{t,s,u} K(\underline{\Delta}_{s,u})\ , \tag{4}$$

where

$$K(\underline{x}) = 1 + \langle \underline{\theta}, \underline{x} \rangle + \langle \underline{x}, \Gamma \underline{x} \rangle$$

with

$$\underline{\theta} = \begin{bmatrix} \theta \\ \eta \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \tau & -1 \\ \gamma & \sigma \end{bmatrix}$$

and

$$F_{t,s,u} = \frac{(u-t)(t-s)}{u(1+s\sigma) + \tau - s\gamma}\ .$$

# Quadratic harness (standard)

It appears that (2), (1) and (4) uniquely determine all the moments of $X$ if they exist.

The moments do exist in all the cases which are known. Moreover, they uniquely determine the distribution of the process. Often finite dimensional distributions are compactly supported.

Therefore the five parameters $\theta$, $\eta$, $\tau$, $\sigma$ and $\gamma$ uniquely determine the distribution of the (standard) quadratic harness. Thus we shall write in this case

$$X \sim QH_0(\underline{\theta}, \Gamma)$$

or

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# Harness

Assume that $X = (X_t)_{t \in T}$, $T = (\alpha, \beta) \subset (0, \infty)$ is a harness. Then

$$E(X_t) = \langle \underline{t},\ \underline{\mu} \rangle$$

where $\underline{t} = (t, 1)^T$ and $\underline{\mu} = (\mu_1, \mu_2)^T$.

If $X$ is square-integrable then

$$Cov(X_s, X_t) = \langle \underline{s},\ \Sigma \underline{t} \rangle\ , \quad s \leq t$$

where

$$\Sigma = \left[ \begin{array}{cc} c_0 & c_1 \\ c_2 & c_3 \end{array} \right] .$$

Here $c_3 \geq 0$, $c_1 > c_2$, $c_0 c_3 > c_2^2$ for positive definiteness.

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# Quadratic harness

Quadratic harness $X$ satisfies (1) and (4) and has general mean $\underline{\mu}$ and general covariance $\Sigma$.

For $\underline{\mu}$ and $\Sigma$ to be compatible with (4) with $\underline{\theta}$ and $\Gamma$ it follows that

$$1 + \langle \underline{\theta}, \underline{\mu} \rangle + \langle \underline{\mu}, \Gamma \underline{\mu} \rangle + \mathit{trace}(\Gamma \Sigma^T) = 0 \ .$$

Then for the quadratic harness $X$ we write

$$X \sim QH(\underline{\mu}, \Sigma, \underline{\theta}, \Gamma)$$

# Transformations

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-degenerate affine function:

$$f(x, y) = [x, y]A + \underline{m}^T \ ,$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \underline{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \ .$$

Let $\phi$ be a Möbius transform defined by $A$, that is

$$\phi(t) = \frac{at + b}{ct + d} \ , \quad t \in S = \phi^{-1}(T) \ .$$

$T$ and $f$ are such that $S$ and $T = \phi(S)$ are intervals (possibly unbounded);
$\phi$ is increasing if $\det(A) > 0$ and decreasing otherwise)

# Transformations

The $f$-transformation of the process $X$ (on $T$) is the process $Y = X^f$ on $S = \phi^{-1}(T)$ defined by

$$Y_t = (ct + d)X_{\phi(t)} + \langle \underline{t}, \underline{m} \rangle \ , \quad t \in S \ ,$$

Examples:
(i) Affine scaling:

$$Y_t = aX_t + bt + c \quad \text{for} \quad f(x, y) = a[x, y] + [b, c] \ .$$

(ii) Linear time-change:

$$Y_t = X_{at} \quad \text{for} \quad f(x, y) = [ax, y] \ .$$

(iii) Time-inversion:

$$Y_t = tX_{1/t} \quad \text{for} \quad f(x, y) = [y, x] \ .$$

Note that

$$(X^f)^g = X^{g \circ f} \ .$$

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# Transformations

Assume $\det(A) > 0$ ($\det(A) < 0$ is similar). Define $\mathcal{F}^f_{s,u} = \mathcal{F}_{\phi(s),\phi(u)}$.

## Theorem

*Let $X \sim QH_0(\theta, \Gamma)$ be a standard quadratic harness on $T$. Let $f$ be an affine function as defined above. Denote*

$$G = 1 - \langle A^{-1}\theta, \underline{m}\rangle + \langle \underline{m}, A^{-1}\Gamma(A^{-1})^T\underline{m}\rangle \ .$$

*If $G \neq 0$ and $Y = X^f$ then*

$$Y \sim QH(\underline{m}, [a, b]^T[c, d], \underline{\theta}_Y, \Gamma_Y)$$

*where*

$$\underline{\theta}_Y = \frac{A^{-1}\underline{\theta} - (\widetilde{\Gamma} + \widetilde{\Gamma}^T)\underline{m}}{G} \quad \text{and} \quad \Gamma_Y = \frac{\widetilde{\Gamma}}{G}$$

*with $\widetilde{\Gamma} = A^{-1}\Gamma(A^{-1})^T$.*

# From QH bridge to standard QH

## Theorem

*Let* $Y \sim QH(\underline{\mu}, \Sigma, \underline{\theta}, \Gamma)$ *on* $(r, v)$ *with*

$$Cov(Y_s, Y_t) = M^2(s - r)(v - t)\ .$$

*(That is* $Y$ *is a QH bridge on* $(r, v)$*.)*
*Let* $g(\tau, \sigma, \gamma) = v(1 + r\sigma) + \tau - r\gamma > 0$*.*

*(Note that*

$$M^2 g(\tau, \sigma, \gamma) = K(\underline{\mu}) = 1 + \langle \underline{\theta}, \underline{\mu} \rangle + \langle \underline{\mu}, \Gamma \underline{\mu} \rangle\ .)$$

*Then* $Y = X^f$*, where* $X \sim QH_0(\underline{\theta}_X, \Gamma_X)$ *on* $(0, \infty)$

$$f(x, y) = \sqrt{\frac{K(\underline{\mu})}{g(\tau, \sigma, \gamma)}}[x - y, vy - rx] + \underline{\mu}^T$$

# From QH bridge to standard QH, cont.

## Theorem (cont.)

*The parameters of $X$ are*

$$\begin{bmatrix} \theta_X \\ \eta_X \end{bmatrix} = \frac{\begin{bmatrix} v[\theta - r\eta + 2\tau\mu_1 - 2r\sigma\mu_2 - (1-\gamma)(\mu_2 - r\mu_1)] \\ -\theta + v\eta - 2\tau\mu_1 + 2\sigma v\mu_2 - (1-\gamma)(v\mu_1 - \mu_2) \end{bmatrix}}{\sqrt{vg(\tau,\sigma,\gamma)K(\underline{\mu})}}$$

*and*

$$\Gamma_X = \begin{bmatrix} \frac{v[\sigma r^2 + (1-\gamma)r + \tau]}{v(r\sigma+1)+\tau-r\gamma} & -1 \\ \frac{v\gamma - r(v\sigma+1)-\tau}{v(r\sigma+1)+\tau-r\gamma} & \frac{\sigma v^2 + (1-\gamma)v + \tau]}{v(r\sigma+1)+\tau-r\gamma} \end{bmatrix}$$

# Example - Brownian bridge

Let $Y = (Y_t)_{t\in[0,1]}$ be a Brownian bridge. Then

$$Y \sim QH\left(\underline{\mu} = \underline{0}, \Sigma = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \underline{\theta} = \underline{0}, \Gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

and thus $Y = W^f$ where $W$ is a Wiener process and $f(x, y) = [x - y, y]$, that is

$$Y_t = (1 - t)W_{t/(1-t)} \, .$$

# Double conditioning of QH and QH bridge

For $Z \sim QH(\underline{\mu}_Z, \Sigma_Z, \underline{\theta}, \Gamma)$ on $(\alpha, \beta)$ define the conditional process

$$Y = (Y_t)_{t\in[r,v]} \overset{d}{=} [(Z_t)_{t\in[r,v]} | Z_r = z_r, Z_v = z_v] \; .$$

Then

$$Y \sim QH\left(\underline{\Delta}_{r,v}(z_r, z_v), M^2 \begin{bmatrix} -rv & v \\ r & -1 \end{bmatrix}, \underline{\theta}, \Gamma\right) \; ,$$

where

$$M = \sqrt{\frac{K(\underline{\Delta}_{r,v}(z_r, z_v))}{g(\tau, \sigma, \gamma)}} \; .$$

# Double conditioning of QH and QH bridge, cont.

Since

$$Cov(Y_s, Y_t) = M^2(v-t)(s-r)$$

the process $Y$ is a QH bridge on $(r, v)$.

Thus $Y = X^f$ where $X \sim QH_0(\underline{\theta}_X, \Gamma_X)$ with $f$ and the parameters $\underline{\theta}_X$ and $\Gamma_X$ as given in the above Theorem with $\underline{\mu} = \underline{\Delta}_{r,v}(z_r, z_v)$.

# Example - Dirichlet QH

Let $Z$ be a standardized Lévy gamma process, that is

$$Z \sim QH_0\left(\begin{bmatrix}\theta\\0\end{bmatrix},\begin{bmatrix}\frac{\theta^2}{4} & -1\\1 & 0\end{bmatrix}\right).$$

Let $Y$ be a gamma bridge (or Dirichlet process), that is

$$Y = (Y_t)_{t\in[0,v]} \overset{d}{=} [(Z_t)_{t\in[0,v]}|Z_v = z_v].$$

Then $Y = X^f$ and

$$X \sim QH_0\left(\frac{2\theta}{\sqrt{v}\sqrt{4v+\theta^2}}\begin{bmatrix}v\\-1\end{bmatrix},\begin{bmatrix}\frac{v\theta^2}{4v+\theta^2} & -1\\\frac{4v-\theta^2}{4v+\theta^2} & \frac{\theta^2}{v(4v+\theta^2)}\end{bmatrix}\right).$$

Note that $\gamma_X = 1 - 2\sqrt{\tau_X\sigma_X}$, $\tau_X = \theta_X^2/4$ and $\sigma_X = \eta_X^2/4$.

# Example - Binomial QH

Let $Z$ be a standardized Poisson process, that is

$$Z \sim QH_0\left(\begin{bmatrix} \theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

Let $Y$ be a conditional process defined by

$$Y = (Y_t)_{t\in[0,v]} \overset{d}{=} [(Z_t)_{t\in[0,v]} | Z_v = n] .$$

Then $Y = X^f$ and

$$X \sim QH_0\left(\frac{1}{\sqrt{n}}\begin{bmatrix} \sqrt{v} \\ -1/\sqrt{v} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

Note that $\gamma_X = 1$, $\tau_X = \sigma_X = 0$, and $\theta_X \eta_X < 0$.

# Gluing

For any quadratic harness $X$, any time $v$, and any $x \in suppX_v$ there exist independent standard QH's $X_-$, $X_+$ and affine functions $f_-$, $f_+$ such that

$$[(X_t)_{t<v}|X_v = x] \stackrel{d}{=} X_-^{f_-} \quad \text{and} \quad [(X_t)_{t>v}|X_v = x] \stackrel{d}{=} X_+^{f_+} .$$

Then we say that $X$ is constructed by gluing together $X_-$ and $X_+$ through $X_v$.

The Wiener process $(W_t)$ arises e.g. from gluing two Wiener processes through $W_1$.

The gamma process arises from gluing together Dirichlet and gamma processes.

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# Gluing with simple gluands

When in such gluing constructions for QH processes the components $X_-$ and $X_+$ are (relatively) simple?

For instance: $q$-Meixner processes - $QH(\theta, 0, \tau, 0, q)$.

This class contains e.g. five Lévy-Meixner processes: Wiener, Poisson, gamma, Pascal, Meixner - $QH(\theta, 0, \tau, 0, 1)$.

# Gluing with simple gluands, cont.

Unexpectedly, except of the Wiener process, only two such constructions are possible:

1. Classical bi-Poisson process- a QH with $\gamma = 1$, $\theta\eta > 0$, $\sigma = \tau = 0$ arises from gluing two Poisson processes through a gamma random variable.

2. Classical bi-Pascal process- a QH with $\gamma = 1 + 2\sqrt{\tau\sigma}$, $\eta\sqrt{\tau} = \theta\sqrt{\sigma}$ arises from gluing two Pascal processes through a second kind beta random variable.

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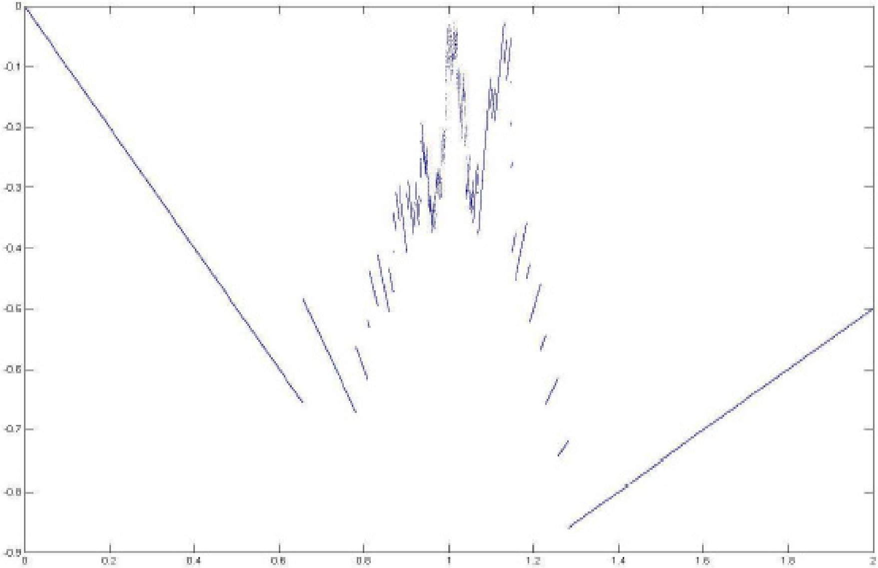
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# Bi-Pascal process, Jamiołkowska (2009), MSc Thesis

Let $a > 0$, $b > 2$

$$h(t) = \frac{(b-1)t}{1-t}\ , t \in [0,1), \quad f(t) = \frac{b-1}{t-1}\ ,\ t > 1$$

Then the family of distributions

$$\mu_{x,s,t} = \begin{cases} NH_{II}^{x}(a+x, b+h(s), h(t)-h(s)) & 0 \le s < t < 1,\ x \in \mathbf{N} \\ Beta_{II}(a+x, b+h(s)) & 0 \le s < t = 1,\ x \in \mathbf{N} \\ NH_{II}(a+x, b+h(s), f(t)) & 0 \le s < 1 < t,\ x \in \mathbf{N} \\ nb(f(t), \frac{1}{1+x}) & s = 1 < t,\ x > 0 \\ NH_{I}(x, f(t), f(s)-f(t)) & 1 < s < t,\ x \in \mathbf{N} \end{cases}$$

satisfies Chapmann-Kolmogorov equations, and thus defines a Markov process $Z = (Z_t)_{t \ge 0}$.

# Bi-Pascal process

Bi-Pascal process $X = (X_t)_{t\geq 0}$ with parameters $a > 0$ and $b > 2$ is defined as

$$X_t = A \begin{cases} (1-t)Z_t - at, & \text{if } t \in [0,1), \\ (b-1)Z_1 - a, & \text{if } t = 1, \\ (t-1)Z_t - a, & \text{if } t > 1 \end{cases},$$

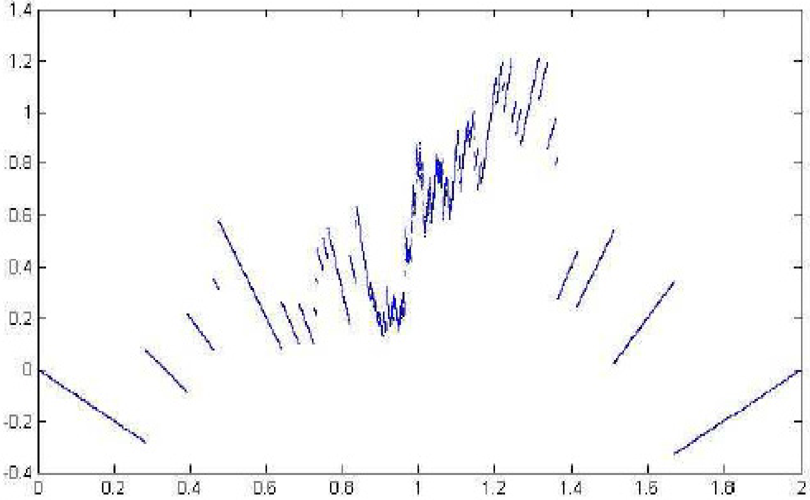
where

$$A = \sqrt{\frac{b-2}{a(a+b-1)}}$$

It appears that $X \sim QH_0$ with parameters

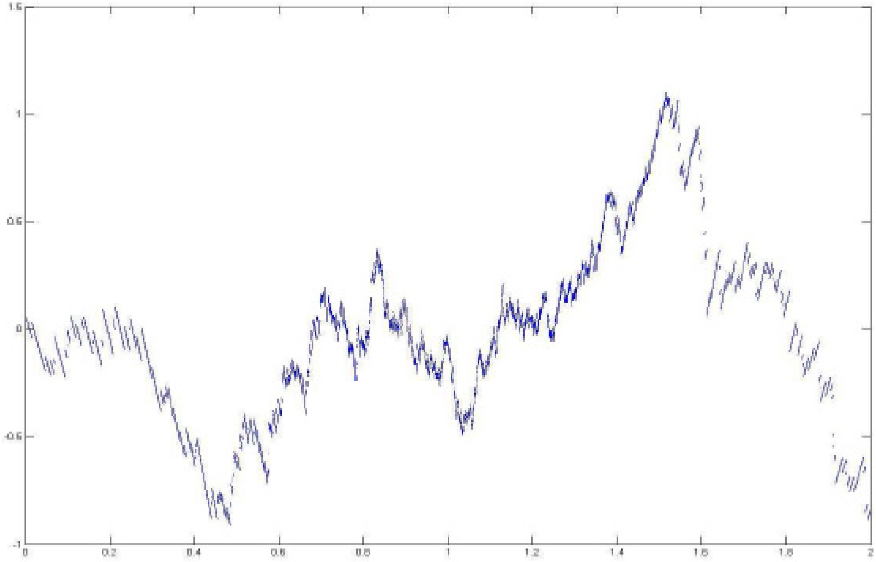
$$\theta = \eta = \frac{2a+b-1}{\sqrt{a(b-2)(a+b-1)}},$$

$$\tau = \sigma = \frac{1}{b-2} \quad \text{and} \quad \gamma = \frac{b}{b-2}$$




$a = 5, \; b = 5$

First family of lines: $A[k - (a + k)t], t \in [0, 1)$.
Second family of lines: $A(kt - a - k)$, $t > 1$.



$a = b = 100$