On subexponentiality of the Lévy measure of the diffusion inverse local time Application to penalization

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1 Introduction

1.1 Notations

– (X_t) denotes a recurrent diffusion on $[0, \infty[$.
– 0 is an instantaneous reflecting barrier.
– Under P_x, (X_t) starts at level x.
– (L_t) is the local time process at 0 of (X_t).

1.2 The goal

Let $h : [0, \infty[\to [0, \infty[$.
We are interested in the asymptotic behaviour of

$$t \mapsto E_x\big(h(L_t)\big) \text{ as } t \to \infty.$$

The Brownian motion case.

Under P_0, L_t is distributed as $\sqrt{t}|G|$ where $G \sim N(0,1)$.
Consequently :

$$E_0\big(h(L_t)\big) \sim \Big(\sqrt{\frac{2}{\pi}} \int_0^{\infty} h(y)dy\Big) \frac{1}{\sqrt{t}}, \quad t \to \infty$$

when $\displaystyle\int_0^{\infty} h(y)dy < \infty$.
In particular, for $h := 1_{[0,l]}$ we have :

$$P_0\big(L_t \leq l\big) \sim \sqrt{\frac{2}{\pi}} \frac{l}{\sqrt{t}}, \quad t \to \infty.$$

The Bessel case.

Suppose that (X_t) is a Bessel process with dimension $\delta \in]0,2[$.

Let $\alpha = 1 - \dfrac{\delta}{2}$

Then

$$L_t \sim t^\alpha L_1 \quad \text{and} \quad \lim_{x\to 0} P_{L_1}(x) = \frac{2^{1-\alpha}}{\Gamma(\alpha)}.$$

As result :

$$E_0\big(h(L_t)\big) \sim \Big(\frac{2^{1-\alpha}}{\Gamma(\alpha)} \int_0^\infty h(y)dy\Big)\frac{1}{t^\alpha}, \quad t \to \infty$$

when $\displaystyle\int_0^\infty h(y)dy < \infty$, and

$$P_0\big(L_t \leq l\big) \sim \frac{2^{1-\alpha}}{\Gamma(\alpha)}\frac{l}{t^\alpha}, \quad t \to \infty.$$

Nota. The Brownian case corresponds to $\delta = 1$, i.e. $\alpha = 1/2$.

2 Subexponential probability distributions

Definition *The (non-degenerated) probability distribution function F on $(0, +\infty)$ is called subexponential if*

$$\lim_{x \to +\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

where $*$ denotes the convolution and $\overline{F}(x) := 1 - F(x)$ the complementary distribution function.

Remarks(Christyakov (1964) and Embrechts and al (1979))
1) If $\overline{F}(x) \sim \dfrac{C}{x^\beta}$, as $x \to \infty$, with $\beta > 0$. Then F is subexponential.
2) Let F be a subexponential distribution. Then for any $y \geq 0$,

$$\lim_{x \to +\infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1.$$

3) The terminology "subexponential" comes from the following property :

$$\lim_{x \to +\infty} e^{\varepsilon x} \overline{F}(x) = +\infty \quad \text{for any} \quad \varepsilon > 0$$

where F is subexponential.

Lemma 1 *Let F be subexponential. Let G be a distribution over $(0, \infty)$ such that $\lim_{x \to \infty} \frac{\overline{G}(x)}{\overline{F}(x)} = c \in (0, \infty)$. Then :*

$$\lim_{x \to \infty} \left(\frac{\overline{F * G}(x)}{\overline{F}(x) + \overline{G}(x)} \right) = 1.$$

Nota. If $G = F$ then we recover the definition.

3 The rate of decay of $P_x(L_t \leq l)$, $t \to \infty$

Let (τ_l) be the right inverse of $t \mapsto L_t$:

$$\tau_l := \inf\{t \geq 0, L_t > l\}, \quad l \geq 0.$$

Recall that $l \mapsto \tau_l$ is non-decreasing, right-continuous, and (τ_l) is a Lévy process. Let ν be its Lévy measure :

$$E\big(e^{-\lambda \tau_l}\big) = \exp\Big\{ -l \int_0^\infty \big(1 - e^{-\lambda x}\big)\nu(dx)\Big\}, \quad \lambda > 0.$$

Theorem (Sato, 1999) *The following are equivalent :*

1) $F(x) := \dfrac{\nu\big(]1, x]\big)}{\nu\big(]1, \infty[\big)}$ *is subexponential.*

2) $P_0\big(\tau_l \geq t\big) \sim l\nu\big(]t, \infty[\big), \quad t \to \infty.$

3) For any l*, the law of* τ_l *is subexponential.*

From now on, we suppose that the conditions of Sato's theorem hold. Let S be the scale function such that $S(0) = 0$ and $S(+\infty) = +\infty$.

Theorem 2 *For any $x \geq 0$ and $l > 0$, we have*

$$P_x\big(L_t < l\big) = P_x\big(\tau_l > t\big) \sim \big(S(x) + l\big)\nu\big(]t, \infty[\big) \quad t \to \infty.$$

Proof

Step 1 Using the strong Markov property we have

$$P_x\big(\tau_l > t\big) = P_x\big(H_0 + \widehat{\tau}_l > t\big)$$

where $H_0 := \inf\{t \geq 0, X_t = 0\}$ and $\widehat{\tau}_l$ is a copy of τ_l independent from H_0.

Let F be the distribution function of τ_l under P_0 and G the distribution function of H_0 under P_x. Then

$$P_x\big(\tau_l > t\big) = \overline{F * G}(t).$$

From Sato's theorem : F is subexponential and

$$\overline{F}(t) = P_0\big(\tau_l > t\big) \sim l\nu\big(]t,\infty[\big), \quad t \to \infty.$$

Suppose for a while that

$$\text{(1)} \qquad \overline{G}(t) = P_x\big(H_0 > t\big) \sim S(x)\nu\big(]t,\infty[\big), \quad t \to \infty.$$

Then :

$$\lim_{t\to\infty} \frac{\overline{G}(t)}{\overline{F}(t)} = \frac{S(x)}{l}.$$

Consequently, Lemma 1 implies

$$\begin{aligned} P_x\big(\tau_l > t\big) = \overline{F * G}(t) &\sim \overline{F}(t) + \overline{G}(t) \\ &\sim \big(S(x) + l\big)\nu\big(]t,\infty[\big) \quad t \to \infty \end{aligned}$$

Step 2 (Proof of (1)). We have the representation formulae

$$\overline{G}(t) = P_x\big(H_0 > t\big) = \frac{1}{\gamma}\int_0^\infty e^{-\gamma t} C(x,\gamma)\Delta(d\gamma)$$

$$\nu\big(]t,\infty[\big) = \frac{1}{\gamma}\int_0^\infty e^{-\gamma t}\Delta(d\gamma)$$

where

$$\int_0^\infty \frac{\Delta(d\gamma)}{\gamma(\gamma+1)} < \infty, \quad \int_0^\infty \frac{\Delta(d\gamma)}{\gamma} = \infty$$

and $C(\cdot,\gamma)$ solves

$$C(x,\gamma) = S(x) - \gamma\int_0^x dS(y)\int_0^y C(y,\gamma)m(dx).$$

($m(dy)$ is the speed measure).
ν subexponential and $C(x,\gamma) \sim S(x)$, $\gamma \to 0$ imply that

$$\overline{G}(t) = P_x\big(H_0 > t\big) \sim S(x)\nu\big(]t,\infty]\big), \quad t \to \infty. \qquad \blacksquare$$

The Bessel case Suppose that (X_t) is a Bessel process with dimension $\delta \in]0,2[$.

Let $\alpha = 1 - \frac{\delta}{2}$

Then $S(x) = \frac{1}{2\alpha} x^{2\alpha}$ and

$$\nu\big(]t,\infty[\big) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} \frac{1}{t^\alpha}, \quad t > 0$$

$$P_x(L_t < l) \sim \frac{2^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{x^{2\alpha}}{2\alpha} + l\right) \frac{1}{t^\alpha} \quad t \to \infty.$$

We recover the result of Roynette, Vallois and Yor (2007).

Note that :

$$m(dx) = 2x^{1-2\alpha} 1_{\{x>0\}} dx, \quad \Delta(d\gamma) = \frac{2^{1-\alpha}}{\Gamma(\alpha)^2} \gamma^\alpha d\gamma$$

$$C(x,\gamma) = \Gamma(\alpha) 2^{(\alpha-2)/2} \gamma^{-\alpha/2} x^\alpha J_\alpha\big(x\sqrt{2\gamma}\big)$$

4 Penalization with the weight $F_t = h(L_t)$

Proposition 3 *Let* $h : [0,\infty[\to [0,\infty[$ *such that* $\int_0^\infty h(y)dy < \infty$. *Let* H *be its primitive :* $H(x) := \int_0^x h(y)dy < \infty$, $x \geq 0$. *Then*

$$M_t^h := S(X_t)h(L_t) + 1 - H(L_t), \quad t \geq 0$$

is a continuous non-negative martingale,

$$M_0^h = 1$$

and

$$M_t \to 0 \quad as \quad t \to \infty.$$

Theorem 4 *Let* $h : [0, \infty[\to [0, \infty[$ *non-increasing with compact support (for instance* $h(x) := \frac{1}{l} 1_{[0,l]}(x)$*). Then, we have a penalization with the weight process* $F_t := h(L_t)$*, i.e. for any fixed* $u > 0$ *:*

$$\lim_{t \to \infty} \frac{E_0\big(1_{\Lambda_u} F_t\big)}{E_0(F_t)} = \lim_{t \to \infty} \frac{E_0\big(1_{\Lambda_u} h(L_t)\big)}{E_0(h(L_t))}$$

$$= E\big(1_{\Lambda_u} M_u^h\big)$$

where $\Lambda_u \in \mathcal{F}_u$*.*

Proof We consider the case $h(x) := \frac{1}{l} 1_{[0,l]}(x)$.

1) According to Sato's theorem :

$$E_0(h(L_t)) = \frac{1}{l} P_0(L_t < l) \sim \nu\big([t, \infty[\big), \quad t \to \infty.$$

2) The Markov property implies that $E_0\big(h(L_t) | \mathcal{F}_u\big) = F(L_u, X_u, t - u)$ where

$$F(a, x, r) = \frac{1}{l} P_x(L_r < l - a).$$

Therefore Theorem 2 implies that

$$F(a,x,t-u) \sim \frac{1}{l}\big(l-a+S(x)\big)\nu\big([t-u,\infty[\big), \quad t\to\infty,$$

and as $t\to\infty$ we have :

$$\frac{E_0\big(h(L_t)|\mathcal{F}_u\big)}{E_0(h(L_t))} \sim \frac{1}{l}\big(l-L_u+S(X_u)\big)1_{\{L_u<l\}}\frac{\nu\big([t-u,\infty[\big)}{\nu\big([t,\infty[\big)}$$

$$\sim M_u^h.$$

To conclude use

$$\frac{E_0\big(1_{\Lambda_u}h(L_t)|\mathcal{F}_u\big)}{E_0(h(L_t))} = E\left(1_{\Lambda_u}\frac{E_0\big(h(L_t)|\mathcal{F}_u\big)}{E_0(h(L_t))}\right)$$

and Sheffe's lemma.

■

Let Q^h be the non-negative probability measure

$$Q^h(\Lambda_u) := E\big(1_{\Lambda_u} M_u^h\big) \quad \Lambda_u \in \mathcal{F}_u.$$

where $h \geq 0$ and verifies $\displaystyle\int_0^\infty h(y)dy = 1$. Then

1. Q^h is a p.m. on $(\Omega, \mathcal{F}_\infty)$
2. $Q^h(L_\infty < \infty) = 1$, and the density of L_∞ under Q^h is h. In particular (X_t) is transient under Q^h.
3. Let $\theta := \sup\{t \geq 0; X_t = 0\}$ (with the convention $\sup \emptyset = 0$). Then
   (a) $Q^h(0 < \theta < \infty) = 1$.
   (b) $\big(X_t, 0 \leq t \leq \theta\big)$ and $\big(X_{t+\theta}, t \geq 0\big)$ are independent.
   (c) Conditionally on $L_\infty = l$, the process $\big(X_t, 0 \leq t \leq \theta\big)$ under Q^h is distributed as $\big(X_t, 0 \leq t \leq \tau_l\big)$ under P_0.

Proof of item 2

$$Q^h(L_u \geq l) = E_0\big(1_{\{L_u \geq l\}} M_u^h\big) = E_0\big(1_{\{\tau_l \leq u\}} M_u^h\big).$$

The stopping theorem implies that

$$E_0\big(1_{\{\tau_l \geq u\}} M_u^h\big) = E_0\big(1_{\{\tau_l \geq u\}} M_{\tau_l}^h\big).$$

But

$$\begin{aligned} M_{\tau_l}^h &= S(X_{\tau_l})h(L_{\tau_l}) + 1 - \int_0^{L_{\tau_l}} h(y)dy \\ &= 1 - \int_0^l h(y)dy = \int_l^\infty h(y)dy. \end{aligned}$$

As a result

$$Q^h(L_u \geq l) = \Big(\int_l^\infty h(y)dy\Big) P_0(\tau_l \leq u).$$

Taking $u \to \infty$ we get :

$$Q^h(L_\infty \geq l) = \int_l^\infty h(y)dy. \qquad \blacksquare$$