Markovian bridges

Weak continuity and pathwise constructions

(Joint work with Loïc CHAUMONT)

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Paul Lévy's Markovian construction of the Brownian bridge

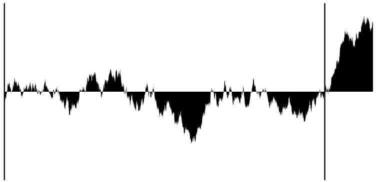
Theorem (P. Lévy)

Let B be a Brownian motion and $g = \sup\{s \leq t : B_s = 0\}$. Then

$$\left(\frac{B_{sg}}{\sqrt{g}}\right)_{s\in[0,1]}$$

has the same law as

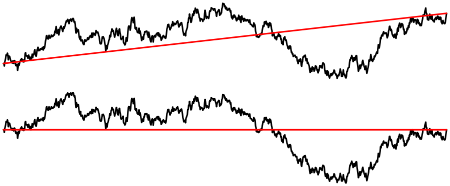
$$(B_s)_{s\in[0,1]} \text{ conditioned on } B_1 = 0.$$



What is meant by the conditioning?

Paul Lévy's Gaussian constructions of the Brownian bridge

- $b_s^{x,y,t} = x + B_s - \frac{s}{t} B_t + \frac{s}{t}(y - x).$
  $x + B_s = b_s^{x,y,t} + \frac{s}{t}(x + B_t - y)$



$$\mathbb{E}_x(F(B_s, s \leq t)\, f(B_t)) = \int \mathbb{P}_x(B_t \in dy)\, f(y)\, \mathbb{E}\left(F\left(b_s^{x,y,t}, s \leq t\right)\right).$$

- $\left(B_{(1-t)/t}\right)_{t \leq 1}$ has law $b^{0,0,1}$.

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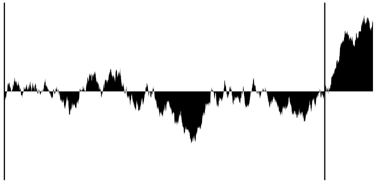

Chaumont's problem for Brownian motion

- Let B be a Brownian motion starting at zero.
- $g_c = \sup\left\{s \leq 1 : B_s = c\sqrt{s}\right\}$.
- Why is $g_c > 0$?
- What is the law of

$$\left(\frac{B_{sg_c}}{\sqrt{g_c}}\right)_{s\in[0,1]}?$$

- Conjecture: That of $B_s, s \leq 1$ conditioned on $B_1 = c$ and $B_0 = 0$.
- Meaning: That of $b^{0,c,1}$.

Yor's Gaussian solution

- $\tilde{B}_t = tB_{1/t}$
- $T = \inf\left\{t \geq 1 : \tilde{B}_t = c\sqrt{t}\right\}$ is a stopping time
- $g_c = \sup\left\{s \leq 1 : B_s = c\sqrt{s}\right\} = 1/T$.
- $X_t = \tilde{B}_{T(1+t)}/\sqrt{T} - c$.
- X is Brownian motion.
- $tX_{(1-t)/t} = B_{tg_c}/\sqrt{g_c} - tc,\ t \leq 1$ has law $b^{0,0,1}$.
- $B_{tg_c}/\sqrt{g_c},\ t \leq 1$ has the same law as $b^{0,c,1}$.

An Markovian proposal for a solution

- Strong Markov property for random times like g_c?
- Yes!!!!
  - L is a backward optional time if
$$\{L > t\} \in \sigma(B_s : s \geq t).$$
  - $\mathscr{L}\left(B^L \,\middle|\, \sigma(B_{L+s} : s \geq 0)\right)$ is $\mathbb{P}^L_{0,B_L}$.
- $B_{g_c} = c\sqrt{g_c}$ by definition of g_c.
- Scaling of bridge laws implies $B_{sg_c}/\sqrt{g_c}$, $s \in [0, 1]$ has law of $b^{0,c,1}$.

Chaumont's general problem

- Let $\mathbb{P}_x$ ($x \geq 0$ or $x \in \mathbb{R}$) be the law of a Markov process started at x.
- Suppose $\mathbb{P}_x$ has the scaling property

$$\text{Law of } X_{ct},\ t \geq 0 \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{c^{1/\gamma}x}.$$

- Let $g_c = \sup\left\{t \leq 1 : X_t = ct^{1/\gamma}\right\}$.

1. Find conditions under which $\mathbb{P}_0(g_c > 0) = 1$.
2. Make sense of the conditional law of X_s, $s \leq t$ given $X_t = y$ under $\mathbb{P}_x$.
3. (Call it $\mathbb{P}^t_{x,y}$.)
4. Is $\mathbb{P}^1_{0,c}$ the law of $X_{sg_c}/g_c^{1/\gamma}$, $s \in [0, 1]$ under $\mathbb{P}_0$?

Answer is (basically) yes if bridges have scaling and backward strong Markov property holds.

The unexpected case of stable Lévy processes

- ξ is a stable Lévy process of index α:

$$\left|\mathbb{E}\left(e^{iu\xi_t}\right)\right| = e^{-Ct|u|^{\alpha}}.$$

- $g_c = \sup\left\{s \leq 1 : \xi_s = ct^{1/\alpha}\right\}$.
- $\mathbb{P}(g_c > 0) \in \{0, 1\}$
- $\mathbb{P}(g_c > 0) = 1$?

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Theorem

If $\alpha > 1$ or $\alpha < 1$ and $c \neq 0$ then $\left(\frac{1}{g_c^{1/\gamma}}\xi_{sg_c}\right)_{s\in[0,1]}$ is a bridge of ξ from 0 to c of length 1.

General construction by approximation

Let $\mathbb{P}_x$ be the law of a Feller process started at x.

Consider the conditional law of $X_s, s \leq t$ given $\rho(X_t, y) < \varepsilon$ under $\mathbb{P}_x$: $\mathbb{P}^t_{x,y,\varepsilon}$.

Theorem

Suppose

$$\mathbb{P}_x(f(X_s)) = \int p_s(x, y)\, f(y)\, \mu(dy)\,.$$

Under the technical conditions

H1

H2

H3

$\mathbb{P}^t_{x,y,\varepsilon}$ *converges weakly as* $\varepsilon \to 0$ *to* $\mathbb{P}^t_{x,y}$ *and* $\left(\mathbb{P}^t_{x,y}\right)_y$ *is a weakly continuous version of the conditional law of* $X_s, s \leq t$ *given* X_t *under* $\mathbb{P}_x$.

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$$\mathbb{P}_x(f(X_s)) = \int p_s(x,y)\, f(y)\, \mu(dy)\,.$$

Under the technical conditions

- H1 $y \mapsto p_s(x,y)$ *is continuous for* $s \in (0,t]$.
- H2 $s \mapsto p_s(x,y)$ *is continuous*
- H3 *The CK equations* $p_t(x,y) = \int p_s(x,z)\, p_{t-s}(z,y)\ \mu(dz)$ *hold.*

$\mathbb{P}^t_{x,y,\varepsilon}$ *converges weakly as* $\varepsilon \to 0$ *to* $\mathbb{P}^t_{x,y}$ *and* $\left(\mathbb{P}^t_{x,y}\right)_y$ *is a weakly continuous version of the conditional law of* $X_s, s \leq t$ *given* X_t *under* $\mathbb{P}_x$.

Applications

1. Lévy processes with integrable characteristic function. (Kallenberg's construction re-obtained.)
2. Bessel processes.
3. Bessel processes with drift.
4. Killed Brownian motion.
5. Some Ornstein-Uhlenbeck processes associated to Lévy processes...

The Backward Strong Markov Property

- Backward optional time $\{L > t\} \in \sigma(X_u : u \geq t)$.
- $\mathscr{F}^L = \sigma(X_{L+t} : t \geq 0)$.
- A version of the conditional law of $X|_{[0,L)}$ given $\mathscr{F}^L$ is $\mathbb{P}^L_{x,X_{L-}}$.
- In particular, past and future are conditionally independent given a backward optional time and the value of the process at it.

Example of conditional independence

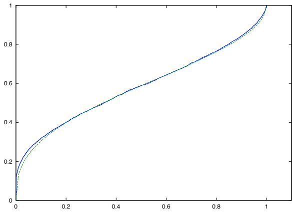
- X is a d-dimensional Bessel process, $X_0 = 0$, $d \geq 3$.
- $L_a = \sup_{t \geq 0 : X_t \leq a}$.
- $L_{b+a} - L_a$ is measurable wrt the process after L_a.
- Hence L is a Sato process.

Remarks on the Brownian case

- B: Brownian movement.
- $g_c = \sup\left\{t \leq 1 : B_t = c\sqrt{t}\right\}$.
- Lévy's first arcsine law: g_0 has $\mathrm{Beta}(1/2, 1/2)$ law.

Question

What is the distribution of g_c when $c \neq 0$?



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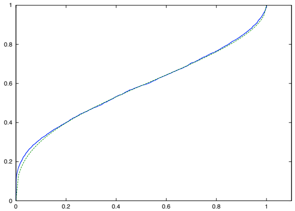

1. $H_q(x) = \int_0^\infty e^{-xz - z^2/2} z^{q-1}\, dz$.
2. $\mathbb{E}((1-g_c)^q) = \frac{\Gamma(2q)}{2qH_{2q}(c)H_{2q}(-c)}$.

Subordinators conditioned to die at a given level

- $\mathbb{P}_x^\alpha$: law of a stable subordinator of index α started at x.
- f_t: density of X_t under $\mathbb{P}_0$.
- transition density:
  $\mathbb{P}_x(X_t \in dy) = p_t(x, y)\ dy = f_t(y - x)\ dy$.
- potential density: $u^\alpha(a) = \int_0^\infty f_t(a)\ dt$.
- $b > 0$ fixed,

$$h_\alpha(x) = \begin{cases} u^\alpha(b - x) & \text{if } x < b \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbb{P}_x^{h_\alpha}|_{\mathscr{F}_t} = \frac{h_\alpha(X_t)}{h_\alpha(x)} \cdot \mathbb{P}_x^\alpha|_{\mathscr{F}_t}$ for $x \leq b$.
- ζ =death time: $\mathbb{P}_x^{h_\alpha}(\zeta < \infty) = 1 = \mathbb{P}_x^{h_\alpha}(X_{\zeta-} = b)$.

Subordinators conditioned to die at a given level

- $L = \sup\{t \geq 0 : X_t \leq b\}$.
- $g = X_{L-}$
- Define Y by

$$Y_t = \begin{cases} \frac{b}{g} X_{t(g/b)^\alpha} & \text{if } t\,(g/b)^\alpha < L \\ \Delta & \text{otherwise} \end{cases}$$

- The law of Y under $\mathbb{P}_0^\alpha$ is $\mathbb{P}_0^{h_\alpha}$.