Application of Malliavin calculus to long-memory parameter estimation for non-Gaussian processes

Ciprian A. Tudor

SAMOS, Université de Paris 1

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Multiple Wiener-Itô integrals

Let $(W_t)_{t\in[0,1]}$ a standard Wiener process.

If $f \in L^2([0,1]^n)$ we define the multiple Wiener integral of with respect to W
Let f be a step function ($f \in \mathcal{S}$), that means

$$f = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} 1_{A_{i_i}\times\ldots\times A_{i_n}}$$

(here $c_{i_1,\ldots,i_n} = 0$ if two indices i_k and i_l are equal and the sets $A_i \in \mathcal{B}([0,1])$ are disjoint)
we define

$$I_n(f) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots i_n} W(A_{i_1})\ldots W(A_{i_n})$$

where $W([a,b]) = W_b - W_a$.

We have that

• the application I_n is an isometry on $\mathcal{S}$, i.e.

$$E\left(I_n(f)I_m(g)\right) = n!\langle f, g\rangle_{L^2([0,1]^n)} \text{ if } m = n$$

and

$$E\left(I_n(f)I_m(g)\right) = 0 \text{ if } m \neq n$$

• the set $\mathcal{S}$ is dense in $L^2([0,1]^n)$

Therefore I_n can be extended to an isometry from $L^2([0,1]^n)$ to $L^2(\Omega)$.

$I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ is the symmetrization of f

Remark : I_n can be viewed as an iterated stochastic Itô integral

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} f(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}$$

Important tool : product formula for multiple stochastic integral

if $f \in L^2([0,1]^n)$ and $g \in L^2([0,1])^m$ are symmetric functions, then

$$I_n(f)I_m(g) = \sum_{l=0}^{m\wedge n} l! \binom{m}{l}\binom{n}{l} I_{m+n-2l}(f \otimes_l g)$$

where the contraction $f \otimes_l g$ belongs to $L^2([0,1]^{m+n-2l})$ for $l = 0, 1, \ldots, m \wedge n$ and it is given by

$$\begin{aligned} & (f \otimes_l g)(s_1, \ldots, s_{n-l}, t_1, \ldots, t_{m-l}) \\ = & \int_{[0,1]^l} f(s_1, \ldots, s_{n-l}, u_1, \ldots, u_l) g(t_1, \ldots, t_{m-l}, u_1, \ldots, u_l) du_1 \ldots du_l \end{aligned}$$

$\hookrightarrow$ One can define multiple integrals with respect to other Gaussian processes (for example the fractional Brownian motion) In this case, we just need to replace the space $L^2([0,1])$ by the canonical Hilbert space of the Gaussian process.

$\hookrightarrow$ In the case of the fBm $\mathcal{H}$ is defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H} \right)$$

We will have the isometry

$$E\left(I_n(f)\right)^2 = n! \|f\|^2_{\mathcal{H}^{\otimes n}}, \quad f \in \mathcal{H}^{\otimes n}$$

.

Recent interest to study the convergence of sequences of the form $I_n(f_k)$ with $k \geq 1$ where $f_k \in L^2([0,1]^n)$ (or $f \in \mathcal{H}^{\otimes n}$).

$\mapsto$ Originally, motivated for example by the study of renormalization of quantities $\int_\varepsilon^1 \left(\frac{B_t}{t}\right)^2 dt$. (papers by Peccati and Yor)

Note that

$$E\int_\varepsilon^1 \left(\frac{B_t}{t}\right)^2 dt = \int_\varepsilon^1 \frac{1}{t} dt = \log(\frac{1}{\varepsilon}) \to_{\varepsilon \to 0} \infty$$

What is the limit of $\int_\varepsilon^1 \left(\frac{B_t}{t}\right)^2 dt - \log \frac{1}{\varepsilon}$ as $\varepsilon \to 0$?
Note also that $\int_\varepsilon^1 \left(\frac{B_t}{t}\right)^2 dt - \log \frac{1}{\varepsilon} = I_2\left(\int_\varepsilon^1 \frac{1_{[0,t]}^{\otimes 2}}{t} dt\right)$

First nice result- By Nualart and Peccati (Annals of Probability '05).

Theorem : Let $F_k = I_n(f_k)$ (f_k symmetric) be a sequence of square integrable random variables in the n th Wiener chaos such that $\mathbf{E}(F_k^2) \to 1$ as $k \to \infty$.

Then the following are equivalent :

i) The sequence $(F_k)_{k\geq 0} = (I_n(f_k))_{k\geq 1}$ converges to the normal law $N(0,1)$.

ii) One has $\mathbf{E}(F_k^4) \to 3$ as $k \to \infty$.

iii) For all $1 \leq l \leq n-1$ it holds $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} \to 0$.

Relation with Malliavin derivatives

Later, Nualart and Ortiz-Latorre prove that the above assertions are equivalent to

$$\|DF_k\|^2_{\mathcal{H}} \to n$$

in $L^2(\Omega)$ as $k \to \infty$, where D is the Malliavin derivative is with respect B.

This is true because

$$E\left(1 - \frac{1}{n}DF_k\right)^2 = (1 - n!\|f_k\|^2_{\mathcal{H}^{\otimes n}})^2 + \sum_{r=1}^{n} c_{n,r}\|f_k \otimes f_k\|^2_{\mathcal{H}^{\otimes 2(n-r)}}$$

Recall : If $f \in \mathcal{H}^{\otimes n}$ is symmetric then we will use the following rule to differentiate in the Malliavin sense

$$D_t I_n(f) = n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0,1].$$

Example : $D_s W_t = 1_{[0,t]}(s)$, $D_s W_t^n = n W_t^{n-1} 1_{[0,t]}(s)$

Let B be a fractional Brownian motion with Hurst parameter $H \in (0,1)$.
Consider the statistics

$$V_N := \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\left| B(i/N) - B((i-1)/N) \right|^2}{N^{-2H}} - 1 \right) \qquad (1)$$

-the study of these variations is equivalent to the study of statistical estimators for the selfsimilarity index
$\mapsto$ actually $V_N = (\log(N)(\hat{H}_N - H)$ where

$$\hat{H}_N = -\frac{\log \frac{1}{N} \sum_{i=1}^{N} \left(X(\frac{i}{N}) - X(\frac{i-1}{N}) \right)^2}{2 \log N}. \qquad (2)$$

It is easy to decompose in chaos the statistics V_N (here A_i denotes $1_{[t_i,t_{i+1}]}$)

$$V_N = N^{2H-1} I_2 \left(\sum_{i=1}^{N} A_i \otimes A_i \right).$$

-clearly the results by Nualart and Peccati can be applied

Results for fBm

Let $H \in (0,1)$ and B be a fractional Brownian motion with parameter H. Consider the standardized quadratic variation V_N. **If** $H \in (0, 3/4)$**,** then

$$F_N := c_H \sqrt{N} V_N \tag{3}$$

converges in law to a standard normal random variable.

If $H \in (3/4, 1)$**,** then

$$\bar{F}_N := N^{2-2H} c_H V_N$$

converges in $L^2(\Omega)$ **to a non-Gaussian limit** (a Rosenblatt random variable with parameter $H_0 = 2H - 1$;)

For $H = \frac{3}{4}$ the limit is still normal but the normalization is different.

For the rate of convergence, see Nourdin-Peccati and Breton -Nourdin.

Estimation for the Rosenblatt process

- it appears originally in the *Non Central Limit Theorem* (see Dobrushin and Majòr ('75) or Taqqu ('75)).
- the Rosenblatt process is selfsimilar with stationary increments
- it has the same covariance as the fBm
- the Rosenblatt process is Holdër continuous of order $\delta < H$
- it is not a Gaussian process ; in fact, it can be written as a double stochastic integral of a two-variable deterministic function with respect to the Wiener process

Recall the representation of the fBm

$$B_t^H = \int_0^t K^H(t,s)dW_s$$

The representation for the Rosenblatt process $(Z_t^H)_{t\in[0,T]}$ is

$$Z_t^H = \int_0^t \int_0^t L_t(y_1,y_2)dW_{y_1}dW_{y_2}$$

where $(W_t, t\in[0,T])$ is a Brownian motion,

$$L_t(y_1,y_2) = d(H)1_{[0,t]}(y_1)1_{[0,t]}(y_2)\int_{y_1\vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du,$$

$H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\left(\frac{H}{2(2H-1)}\right)^{-\frac{1}{2}}$.

Next step : try to estimate the parameter of the Rosenblatt process
Consider the variations given by

$$V_N = \frac{1}{N}\sum_{i=1}^{N} \frac{\left(Z(\frac{i}{N}) - Z(\frac{i-1}{N})\right)^2}{\mathbf{E}\left(Z(\frac{i}{N}) - Z(\frac{i-1}{N})\right)^2} - 1$$
$$= N^{2H-1}\sum_{i=1}^{N}\left[\left(Z(\frac{i}{N}) - Z(\frac{i-1}{N})\right)^2 - N^{-2H}\right].$$

$$I_2(f)I_2(f) = I_4(f \otimes f) + 4I_2(f \otimes_1 f) + 2I_0(f \otimes_2 f);$$

we set

$$A_i := L_{\frac{i}{N}} - L_{\frac{i-1}{N}};$$

we can thus write

$$\left(Z(\frac{i}{N}) - Z(\frac{i-1}{N}) \right)^2 = (I_2(A_i))^2$$
$$= I_4(A_i \otimes A_i) + 4I_2(A_i \otimes_1 A_i) + 2I_0(A_i \otimes_2 A_i)$$

This implies that the 2-variation is decomposed into a 4th chaos term and a 2nd chaos term :

$$V_N = N^{2H-1} \sum_{i=1}^{N} \left(I_4(A_i \otimes A_i) + 4 I_2(A_i \otimes_1 A_i) \right)$$
$$:= T_4 + T_2.$$

T_4 is an element of the fourth Wiener chaos
T_2 is in the second Wiener chaos

A detailed study of the two terms above will shed light on some interesting facts :

• if $H \leq \frac{3}{4}$ the term T_4 continue to exhibit "normal" behavior (renormalized -by $\sqrt{N}$! - it converges in law to a Gaussian distribution),

• while the term T_2, which turns out to be dominant, never converges to a Gaussian law. One can say that the second Wiener chaos portion is "ill-behaved" ;

We will obtain :

if $X = Z$ and $H \in (1/2, 1)$, then

$$N^{1-H} V_N / (4d(H))$$

converges in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$;

so does $\frac{N^{1-H}}{2d(H)} \log(N) \ (\hat{H}_N - H)$;

The proof is based on the estimation of chaos ; for example, for the term T_4, we proved above that $\lim_{N\to\infty} \mathbf{E}(G_N^2) = 1$ for $H < 3/4$ where

$$G_N := \sqrt{N} N^{2H-1} e_{1,H}^{-1/2} I_4 \left(\sum_{i=1}^{N} A_i \otimes A_i \right) \tag{4}$$

the Malliavin derivative of G_N is

$$D_r G_N = \sqrt{N} N^{2H-1} e_{1,H}^{-1/2} 4 \sum_{i=1}^{N} I_3(A_i \otimes A_i)(\cdot, r)$$

and its norm is

$$\|DG_N\|^2_{L^2[0,1]} = N^{4H-1} 16 e_{1,H}^{-1} \sum_{i,j=1}^{N} \int_0^1 dr I_3(A_i \otimes A_i)(\cdot, r) I_3(A_j \otimes A_j)(\cdot, r).$$

The product formula gives

$$\|DG_N\|^2_{L^2[0,1]} = N^{4H-1} 16 e^{-1}_{1,H} \sum_{i,j=1}^{N} \int_0^1 dr \, [I_6 ((A_i \otimes A_i)(\cdot, r) \otimes (A_j \otimes A_j)(\cdot,$$

$$+9I_4 ((A_i \otimes A_i)(\cdot, r) \otimes_1 (A_j \otimes A_j)(\cdot, r))$$

$$+9I_2 ((A_i \otimes A_i)(\cdot, r) \otimes_2 (A_j \otimes A_j)(\cdot, r))$$

$$+3! I_0 ((A_i \otimes A_i)(\cdot, r) \otimes_3 (A_j \otimes A_j)(\cdot, r))]$$

$$:= J_6 + J_4 + J_2 + J_0.$$

Other variations

How to avoid the non-centrality ?

- use "longer filters" (replace the increment $X_{i+1} - X_i$ by the increment $X_{i+1} - 2X_i + X_{i-1}$)

-use other estimators (like wavelet -type estimators)

-use "adjusted variations"

Longer Filters

In the case of the fBm with $H > 3/4$ recall that $N^{2-2H} V_N$ converges in law to a non-Gaussian limit.

This non-centrality can be avoided by using higher order increments; for example replace increment $B_{\frac{i+1}{N}} - B_{\frac{i}{N}}$ by the increment $B_{\frac{i+1}{N}} - 2B_{\frac{i}{N}} + B_{\frac{i-1}{N}}$.

Then for any $H \in (0,1)$ it holds that

$V_N = \frac{1}{N} \sum_{i=0}^{N-1} \left[\frac{\left(B_{\frac{i+1}{N}} - 2B_{\frac{i}{N}} + B_{\frac{i-1}{N}} \right)^2}{E\left(B_{\frac{i+1}{N}} - 2B_{\frac{i}{N}} + B_{\frac{i-1}{N}} \right)^2} - 1 \right]$ renormalized by $\sqrt{N}$

converges to a normal limit.

What happens if we use the longer filters for the Rosenblatt process ?

again the variation V_N can be decomposed into two terms : T_4 in the Wiener chaos of order 4 and T_2 in the Wiener chaos of order 2.

• for every $H \in (\frac{1}{2}, 1)$ the term T_4 is "good" (it converges to a normal distribution)

• for every $H \in (\frac{1}{2}, 1)$ the term T_2 converges to a non-Gaussian limit

• the term T_2 gives again the limit of V_N

Conclusion : the use of the longer filters affects only the behavior of T_4 but it does not affect the limit of V_N.

Using adjusted variations

V_N does not converge to the normal law. But this statistic, which can be written as $V_N = T_4 + T_2$ has a small *normal part*, which is given by the asymptotics of the term T_4.
Therefore, $V_N - T_2$ will converge (under suitable scaling) to the Gaussian distribution. Of course, the term T_2, which is an iterated stochastic integral, is not practical because it cannot be observed. But, replacing it with its limit $Z(1)$ (this *IS* observed) one can defined an adjusted version of the statistics V_N that converges, after standardization, to the standard normal law.

$$\begin{aligned} V_N - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1) &= V_N - T_2 + T_2 - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1) \\ &= T_4 + \frac{\sqrt{c_{3,H}}}{N^{1-H}} \left[\frac{N^{1-H}}{\sqrt{c_{3,H}}} T_2 - Z(1) \right] := T_4 + U_2. \end{aligned} \tag{5}$$

Theorem

Let $(Z(t), t \in [0,1])$ be a Rosenblatt process with selfsimilarity parameter $H \in (1/2, 2/3)$ and let previous notations for constants prevail. Then, the following convergence occurs in distribution :

$$\lim_{N\to\infty} \frac{\sqrt{N}}{\sqrt{e_{1,H} + f_{1,H}}} \left[V_N - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1) \right] = \mathcal{N}(0,1).$$

Wavelets : the basic idea

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function with support included in the interval $[0,1]$. Assume that there exists an integer $Q \geq 1$ such that ψ has Q *zero moments*

$$\int_{\mathbb{R}} t^p \psi(t)dt = 0 \text{ for } p = 0, 1, \ldots, Q-1 \tag{6}$$

and

$$\int_{\mathbb{R}} t^Q \psi(t)dt \neq 0.$$

-in general an increasing number of vanishing moments brings more regularity.

For a stochastic process $(X_t)_{t\in[0,N]}$ and for a "scale" $a \in \mathbb{N}^*$ we define its wavelet coefficient by

$$\begin{aligned} d(a,i) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{t}{a} - i\right) X_t dt \\ &= \sqrt{a} \int_0^1 \psi(x) X_{a(x+i)} dx, \ i = 1, 2, \ldots, [N/a] - 1. \end{aligned}$$

- $\psi\left(\frac{t}{a} - i\right)$ "dilatations" and "translations" of the function ψ.
-$\psi(t/a)$ is a dilatation of ψ by the factor a and $\psi(t-i)$ is a translation to the right by i units.

Let us set

$$\tilde{d}(a,i) = \frac{d(a,i)}{(E(d(a,i))^2)^{\frac{1}{2}}}$$

and

$$V_N(a) = \frac{1}{N_a} \sum_{i=1}^{N_a} \left(\tilde{d}(a,i)^2 - 1 \right).$$

We will study the behavior of the sequence $V_N(a)$ when $N \to \infty$.

Result in the case of the fBm (X is the fBm)

Theorem

If $Q > 1$ *and* $H \in (0,1)$ *or if* $Q = 1$ *and* $H \in (0,3/4)$*, for all* $a > 0$*,*

$$\left(\sqrt{\frac{N}{a}}\, V_N(a)\right) \to_N \mathcal{N}(0\,,\, L_1(H)).$$

Idea : another possibility to avoid the non-centrality of the limit is to use a bigger number of vanishing moments.

Wavelets and Rosenblatt

In this section,assume that Z^H is a Rosenblatt process with selfsimilarity order H. In this case, the wavelet coefficient can be written as

$$\begin{aligned} d(a,i) &= \sqrt{a}\int_0^1 \psi(x) Z^H_{a(x+i)}dx \\ &= \sqrt{a}\int_0^1 \psi(x)dx \left(\int_0^{a(x+i)} \int_0^{a(x+i)} L_{a(x+i)}(y_1,y_2)dW_{y_1}dW_{y_2} \right) \\ &:= I_2^W\left(g_{a,i}(\cdot)\right) \end{aligned}$$

where I_2^W denotes the multiple integral or order 2 with respect to the Wiener process W and

$$g_{a,i}(y_1,y_2) = \sqrt{a}\int_{\frac{y_1\vee y_2}{a}-i}^1 dx\psi(x)\left(\int_{y_1\vee y_2}^{a(x+i)} \frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du\right)$$

We will obtain

$$d^2(a,i) = I_4\left(g_{a,i}^{\otimes 2}\right) + 4I_2\left(g_{a,i}\otimes_1 g_{a,i}\right) + 2\|g_{a,i}\|^2_{L^2[0,1]^2}$$

and noting that

$$Ed(a,i)^2 = E\left(I_2(g_{a,i})\right)^2 = 2\|g_{a,i}\|^2_{L^2[0,1]^2} = a^{2H+1}C_\psi(H)$$

we obtain the following decomposition for the statistic $V_N(a)$:

$$\begin{aligned} V_N(a) &= a^{-2H-1}C_\psi(H)^{-1}\frac{1}{N_a}\left[\sum_{i=1}^{N_a} I_4\left(g_{a,i}^{\otimes 2}\right) + \sum_{i=1}^{N_a} 4I_2\left(g_{a,i}\otimes_1 g_{a,i}\right)\right] \\ &:= T_4 + T_2. \end{aligned}$$

Behavior of T_4

Proposition

if $Q > 1$ and $H \in (\frac{1}{2}, 1)$ or if $Q = 1$ then for $H \in (\frac{1}{2}, \frac{3}{4})$, then

$$\frac{N}{a} E(T_4^2) \to_N 3\,\ell_1(1,1,H) \tag{7}$$

and, if $Q = 1$ and $H \in (\frac{3}{4}, 1)$ then

$$\left(\frac{N}{a}\right)^{4-4H} E(T_4^2) \to_N 3\,\ell_2(H). \tag{8}$$

The estimation of the term T_2 :

$$
\begin{aligned}
(g_{a,i} \otimes_1 g_{a,i})(y_1, y_2) &= aa(H)^{-1} \int_{\frac{y_1 \vee y_2}{a} - i}^{1} \int_{\frac{y_1 \vee y_2}{a} - i}^{1} dx dx' \psi(x) \psi(x') \\
&\times \int_{y_1}^{a(x+i)} \frac{\partial K^{H'}}{\partial u}(u, y_1) du \int_{y_2}^{a(x'+i)} \frac{\partial K^{H'}}{\partial v}(v, y_2)
\end{aligned}
$$

We compute now the mean square of the term T_2. Note that

$$
T_2 = I_2(h_N(a))
$$

with

$$
h_N(a) = 4a^{-2H-1} C_{\psi}(H)^{-1} \frac{1}{N_a} \sum_{i=1}^{N_a} g_{a,i} \otimes_1 g_{a,i}
$$

Result on the normalization : the asymptotic behavior of the term T_2 depends on H and, surprisingly and contrary to the Gaussian case, it is not influenced by the number Q of vanished moment of ψ. Thus for any $Q \geq 1$

$$\mathbf{E}\left[\left(\frac{N}{a}\right)^{1-H} T_2\right]^2 \to_N C_{T_2}^2(H).$$

-the term T_2 is dominant

In conclusion the term T_2 converges as follows

Theorem

Let $(Z_t^H)_{t\geq 0}$ be a Rosenblatt process. Then, for any $Q \geq 1$ and $H \in (\frac{1}{2}, 1)$, there exists a Rosenblatt random variable R_1^H with self-similarity order H such as

$$C_{T_2}^{-1}(H)\left(\frac{N}{a}\right)^{1-H} T_2 \rightarrow_N Z_1^H \implies C_{T_2}^{-1}(H)\left(\frac{N}{a}\right)^{1-H} V_N(a) \rightarrow_N Z_1^H,$$

References

(with Frederi Viens) "Variations and estimators for the selfsimilarity index through Malliavin calculus"

(with Jean-Marc Bardet) "Hurst index estimation for the Rosenblatt process : wavelets and chaos expansion"

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