A multiple stochastic integral criterion for almost sure central limit theorems

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OUTLINE

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What is an ASCLT?

Let $(\Omega, \mathfrak{F}, P)$ be a probability space.

Usual CLT: $\{X_i\}$ iid, $\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1$,

$$S_n = n^{-1/2}(X_1 + \cdots + X_n).$$

Then $\forall x \in \mathbb{R}$,

$$P[S_n \leq x] \longrightarrow P[N \leq x], \qquad N \sim \mathcal{N}(0,1)$$

or

$$\mathbb{E}1_{\{S_n \leq x\}} \longrightarrow \mathbb{E}1_{\{N \leq x\}}.$$

It is unrealistic to expect $1_{\{S_n \leq x\}} \longrightarrow 1_{\{N \leq x\}}$ a.s. However we can try smoothing through Cesaro summation. Recall:

$$\lim_{n \longrightarrow \infty} a_n = a \qquad \Rightarrow \qquad \lim_{n \longrightarrow \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} a_k = a,$$

and observe that the second limit can hold while the first does not.

However, one still does not have:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{\{S_k \le x\}} \longrightarrow 1_{\{N \le x\}}$$

a.s.

What holds is the following, which is called an "almost sure central limit theorem":

Theorem (ASCLT) (Lévy 1937), (Lacey & Phillip 1990):

For all, $x \in \mathbb{R}$, almost surely,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{\{S_k \le x\}} \longrightarrow \mathbb{E} 1_{\{N \le x\}} = P[N \le x]$$

General definition of an ASCLT

Definition (ASCLT): Let $G_n \xrightarrow{\text{law}} G$ as $n \to \infty$. The sequence $\{G_n\}$ is said to satisfy an ASCLT if, for any x in the continuity set of G,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{\{G_k \le x\}} = P[G \le x], \ a.s.$$

or, equivalently, for any $\varphi : \mathbb{R} \to \mathbb{R}$, continuous and bounded,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) = \mathbb{E}\varphi(G), \ a.s.$$

Example: If $G \sim \mathcal{N}(0,1)$, then the ASCLT says that outside a P-null set,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\varphi(G_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{1}{2}x^2} dx.$$

Remarks:

- An ASCLT involves not only the marginal distributions but also the finite-dimensional distributions.

- ASCLT $\not\Rightarrow$ CLT (Berkes, Dehling & Móri, 1991)

- CLT $\not\Rightarrow$ ASCLT: use the Skorokhod representation.

  Suppose $G_n \xrightarrow{\text{law}} G$. Then there is a probability space and random variables $G_n^\star \stackrel{\text{law}}{=} G_n$, $G^\star \stackrel{\text{law}}{=} G$ such that $G_n^\star \xrightarrow{\text{a.s.}} G^\star$.

  Now use the $G_k^\star$ variables instead of the G_k. Then for any $\varphi : \mathbb{R} \to \mathbb{R}$, continuous and bounded, $\varphi(G_k^\star) \xrightarrow{\text{a.s.}} \varphi(G^\star)$ as $k \to \infty$. Thus, almost surely,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k^\star) = \varphi(G^\star) \neq \mathbb{E}\varphi(G^\star).$$

  So the $G_k^\star$ do not satisfy an ASCLT, even though $G_n^\star \xrightarrow{\text{law}} G^\star$.

Ibragimov and Lifshits (2000) criterion for ASCLT

Proposition 1 *Let* $G_n \xrightarrow{\text{law}} G$. *Set*

$$\Delta_n(t) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\Big(e^{itG_k} - E(e^{itG})\Big).$$

If

$$\sup_{|t|\le T} \sum_n \frac{\mathbb{E}|\Delta_n(t)|^2}{n \log n} < \infty \quad \text{for all } T > 0,$$

then, $\{G_n\}$ *satisfies an ASCLT.*

Note: The G_n may be dependent. Also $\Delta_n(t)$ involves $(G_1, \cdots, G_n)$.

Case where the G_n are Gaussian

Theorem 1 *Let* $G_n \sim \mathcal{N}(0,1)$. *If*

<u>*Condition C1:*</u>

$$\sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{\left|E(G_k G_l)\right|}{kl} < \infty$$

holds, then $\{G_n\}$ *satisfies an ASCLT.*

Proof (direct application of IL Criterion)

$$\begin{aligned} E|\Delta_n(t)|^2 &= \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{1}{kl} \left[E\left(e^{it(G_k - G_l)}\right) - e^{-t^2} \right] \\ &= \frac{1}{\log^2 n} \sum_{k,l=1}^{n} \frac{e^{-t^2}}{kl} \left(e^{E(G_k G_l) t^2} - 1 \right) \\ &\leq \frac{T^2 e^{T^2}}{\log^2 n} \sum_{k,l=1}^{n} \frac{\left|E(G_k G_l)\right|}{kl}. \end{aligned}$$

Therefore Condition (C1) implies the ASCLT.

□

Wiener chaos

A Gaussian random variable G_k belongs to the Wiener chaos of order $q = 1$, that is, it can be represented as:

$$G_k = \int_{\mathbb{R}} f_k(x)\ dW(x) = I_1(f_k),$$

where W is a randomly scattered Gaussian random measure with non-atomic control measure μ and where $\|f_k\|^2_{L^2(R)} = \int_{\mathbb{R}} |f_k(x)|^2 d\mu(x) < \infty$.

We shall now suppose that G_k belongs to the Wiener chaos of order $q > 1$, ie,

$$G_k = \int'_{\mathbb{R}^q} f_k(x_1, \cdots, x_q)\, dW(x_1) \cdots dW(x_q) = I_q(f_k)$$

with f_k symmetric,

$$\|f_k\|^2_{L^2(\mathbb{R}^q)} = \int_{\mathbb{R}^q} |f_k(x_1, ..., x_k)|^2 d\mu(x_1) ... d\mu(x_k) < \infty$$

and

$$\mathbb{E}G_k^2 = \mathbb{E}I_q(f_k)^2 = q! \|f_k\|^2_{L^2(\mathbb{R}^q)}.$$

Relating $I_q(f)$ to $I_1(g)$

We now need to relate in some way $I_q(f)$ to $I_1(g) = N \sim \mathcal{N}(0,1)$.

Proposition 2 *(Nourdin & Peccati 2007)*

Let $q \geq 2$, $f \in L^2(R^q)$, *normalized:* $q!\|f_k\|^2_{L^2(R^q)} = 1$. *and* $N \sim \mathcal{N}(0,1)$. *Then, for all Lipschitz functions* $h : \mathbb{C} \to \mathbb{R}$ *with bound coefficient 1:*

$$|h(x) - h(y)| \leq |x - y|, \quad x, y \in \mathbb{C},$$

we have

$$\Big| E[h(I_q(f))] - E[h(N)] \Big|$$

$$\leq \sqrt{E\left[\left(1 - \frac{1}{q}\|D[I_q(f)]\|^2_{L^2(\mathbb{R})}\right)^2\right]}$$

where D denotes the Malliavin derivative.

The Malliavin derivative D

For $f \in L^2(\mathbb{R}^q)$ symmetric,

$$DI_q(f) = qI_{q-1}f(\cdot, x)$$

$$= \int'_{\mathbb{R}^{q-1}} f(x_1, \cdots, x_{q-1}, x)\ dW(x_1) \cdots dW(x_{q-1})$$

so that the quantity that appears in the proposition is nothing else than

$$\|D[I_q(f)]\|^2_{L^2(\mathbb{R})} = q^2 \int_{\mathbb{R}} (I_{q-1}(f(\cdot, x))^2 \mu(dx)),$$

which is a random variable.

Note: To get a feeling, consider the case $q = 1$. Then $DI_1(f) = 1I_0(f) = f = f(x)$, and since $\|f\|^2 = 1$, the proposition gives

$$0 = \Big| E[h(N)] - E[h(N)] \Big|$$

$$\leq \sqrt{E\left[\left(1 - \frac{1}{q}\|D[I_q(f)]\|^2_{L^2(\mathbb{R})}\right)^2\right]} = 0.$$

ASCLT for the chaos case

Theorem 2 *Let* $G_n = I_q(f_n), q \geq 2$, *normalized, i.e.* $q!\|f_k\|^2_{L^2(R^q)} = 1$, *with*

$$G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Suppose:

- *Condition C1:*

$$\sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{\left|E(G_k G_l)\right|}{kl} < \infty,$$

- *If* $q \geq 2$, *suppose in addition:*

  Condition C2: $\forall r = 1, \cdots, q-1,$

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{L^2(\mathbb{R}^{(2q-2r)})} < \infty,$$

  where $f_k \otimes_r f_k$ *denotes the contraction of order* r.

Then $\{G_n\}$ *satisfies an ASCLT.*

The contraction operator

$f_k \otimes_r f_k$ denotes the contraction of order r: each symmetric f_k is a function of q variables. Identify r variables in both functions f_k and integrate over them, to get a function of $2q-2r$ variables.

$\|f_k \otimes_r f_k\|_{L^2(\mathbb{R}^{(2q-2r)})}$ is the norm of this function.

Examples: Let f be a function of p variables and g a function of q variables.

- If $r = p = q$, then $f \otimes_r g = \int fg$ (scalar)

- If $r = 0$, then $f \otimes_0 g = fg$ (function of $p+q$ variables).

Fractional Brownian motion

We want to apply our results on ASCLT to the variations of FBM.

Definition: Standard fractional Brownian motion (FBM)

$$B^H = (B_t^H)_{t \geq 0},$$

with Hurst parameter

$$0 < H < 1$$

is a centered Gaussian process with continuous paths such that

$$E[B_t^H B_s^H] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big), \qquad s, t \geq 0.$$

Properties of FBM

- The process B^H is H-self-similar: for any $a > 0$, the finite-dimensional distributions of B^H_{at} are the same as those of $a^H B^H_t$.

- Its variations

$$Y_k = B^H_{k+1} - B^H_k, \quad k \geq 0,$$

form a stationary dependent sequence with covariance $\rho(r)$, $r \in \mathbb{Z}$, equal to

$$E[Y_k Y_{k+r}] = \frac{1}{2}\Big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\Big),$$

which behaves asymptotically as

$$\rho(r) \sim H(2H-1)|r|^{2H-2} \quad \text{as } |r| \to \infty.$$

Application of our ASCLT theorems

We are now going to apply our ASCLT theorems,

- to the Gaussian case:

$$G_n = \frac{1}{n^H} \sum_{k=0}^{n-1} (B_{k+1}^H - B_k^H), \quad n \geq 1,$$

- to the chaos case:

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H), \quad n \geq 1, \quad q \geq 2$$

  in cases where the limit distribution is Gaussian.

The Gaussian case

Theorem 3 *Let* $0 < H < 1$ *and*

$$G_n = \frac{1}{n^H} \sum_{k=0}^{n-1} (B_{k+1}^H - B_k^H) = \frac{B_n^H}{n^H}, \qquad n \geq 1.$$

Then $\{G_n\}$ *satisfies the ASCLT.*

Proof: Clearly $G_n \overset{\text{law}}{=} G$.

- Show that the Condition C1 (the Gaussian criterion) is satisfied.

- Two cases: $H < 1/2$, $H \geq 1/2$.

The chaos case

Let

$$G_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H), \quad n \geq 1,$$

where

- $q \geq 2$
- $1/2 < H < 1 - \frac{1}{2q}$
- σ_n is such that $\mathbb{E} G_n^2 = 1$.

Facts: • $\sigma_n \to \sigma_\infty \in (0, \infty)$, so the correct normalization is indeed $\sqrt{n}$.

• $G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0,1)$ as $n \to \infty$. (Dobrushin and Major, 1979)

Theorem 4 $\{G_n\}$ *satisfies an ASCLT.*

Steps in the proof:

- Express G_n as $I_q(f_n)$

- Use the fact that for $r = 1, \cdots, q-1$,

$$\|f_k \otimes_r f_k\|_{L^2(\mathbb{R}^{(2q-2r)})} \leq Ck^{-\alpha},$$

  where C and α depend on q and H but not on r and k (Breton and Nourdin, 2008).

- Apply our theorem on chaos of order q, checking Conditions C1 and C2.

Observations in the case where the limit is not Gaussian

Let

$$G_n = \frac{1}{\sigma_n n^{H'}} \sum_{k=0}^{n-1} H_q(B^H_{k+1} - B^H_k), \quad n \geq 1,$$

where

- $q \geq 2$
- $1 - \frac{1}{2q} < H < 1$
- $H' = (H-1)q + 1 \in (1/2, 1)$
- σ_n is such that $\mathbb{E}G_n^2 = 1$.

We have no results in this case about ASCLT.

Observations

- $\sigma_n \to \sigma_\infty \in (0,\infty)$, so the correct normalization is indeed $n^{H'}$.

- $G_n \xrightarrow{\text{law}} I_q(g)$, where

$$I_q(g) = C \int'_{\mathbb{R}^q} \left[\int_{[0,1]^q} \prod_{j=1}^{q} (s_j - u_j)_+^{H-3/2} d^q u \right] d^q W$$

(Taqqu, 1979; Dobrushin and Major, 1979).

- The fact that $G_n \xrightarrow{\text{law}} G$, does not guarantee an ASCLT.

Counterexample

Consider, as before,

$$G_n = \frac{1}{\sigma_n n^{H'}} \sum_{k=0}^{n-1} H_q(B^H_{k+1} - B^H_k), \qquad n \geq 1,$$

where $q \geq 2$, and let

$$G^*_n = \frac{1}{\sigma_n n^{H'}} \sum_{k=0}^{n-1} H_q\left(n^H(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}})\right), \qquad n \geq 1.$$

Then

$$G^\star_n \overset{\text{law}}{=} G^\star_n$$

for each $n \geq 1$. BUT:

Proposition 3 *As $n \to \infty$, there is a process G^* such that $G^*_n \longrightarrow G^\star$ in $L^2(\Omega)$ and a.s., and therefore:*

- *$G^\star_n$ is the corresponding Skorokhod representation of G_n*
- *G^*_n does not verify an ASCLT.*

Remark: The fact that $G_n^\star$ does not verify an ASCLT does not imply that G_n will not satisfy one. This is because an ASCLT

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) = \mathbb{E}\varphi(G), \ a.s.$$

involves not only the marginal distributions but also the finite-dimensional distributions, and while

$$G_n \overset{\text{law}}{=} G_n^\star$$

one has

$$(G_1, \cdots, G_n) \overset{\text{law}}{\neq} (G_1^\star, \cdots, G_n^\star).$$

SUMMARY

We established conditions for a sequence G_k to satisfy an ASCLT, ie for any $\varphi : \mathbb{R} \to \mathbb{R}$, continuous and bounded,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \varphi(G_k) = \mathbb{E}\varphi(G), \ a.s.$$

The G_k we considered are multiple Wiener integrals:

$$G_k = \int_{\mathbb{R}^q}' f_k(x_1, \cdots, x_q)\, dW(x_1) \cdots dW(x_q) = I_q(f_k)$$

We distinguish between two cases: $q = 1$ and $q \geq 2$. Our main result is:

Theorem 5 *Let* $G_n = I_q(f_n), q \geq 2$*, normalized, i.e.* $q!\|f_k\|^2_{L^2(R^q)} = 1$*, with*

$$G_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Suppose:

- *Condition C1:*

$$\sum_{n=2}^{\infty} \frac{1}{n \log^3 n} \sum_{k,l=1}^{n} \frac{\left|E(G_k G_l)\right|}{kl} < \infty,$$

- *If* $q \geq 2$*, suppose in addition:*

  Condition C2: $\forall r = 1, \cdots, q-1,$

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \sum_{k=1}^{n} \frac{1}{k} \|f_k \otimes_r f_k\|_{L^2(\mathbb{R}^{(2q-2r)})} < \infty,$$

  where $f_k \otimes_r f_k$ *denotes the contraction of order* r*.*

Then $\{G_n\}$ *satisfies an ASCLT.*

We applied this result to:

$$G_n = \frac{1}{\sigma_n \sqrt{n}} \sum_{k=0}^{n-1} H_q(B_{k+1}^H - B_k^H), \quad n \geq 1,$$

when

- $q = 1$
- $q \geq 2$ and $G_n \xrightarrow{\text{law}} N \sim \mathscr{N}(0,1)$ as $n \to \infty$.

We showed in both cases that

$\{G_n\}$ satisfies an ASCLT.