Noncolliding Processes and Random Matices

Hideki TANEMURA (Chiba University)
joint work with **Makoto KATORI (Chuo University)**

Selfsimilar processes and their applications

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$N \times N$ Hermitian matrix valued process $(N \in \mathbb{N})$

$$M(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) & \cdots & M_{1N}(t) \\ M_{21}(t) & M_{22}(t) & \cdots & M_{2N}(t) \\ & & \cdots & \\ M_{N1}(t) & M_{N2}(t) & \cdots & M_{NN}(t) \end{pmatrix}$$

with $M_{\ell k}(t) = M_{k\ell}(t)^{\dagger}$.

We consider the case

$$M_{k\ell}(t) = B^{\mathrm{R}}_{k\ell}(t) + \sqrt{-1}B^{\mathrm{I}}_{k\ell}(t), \quad 1 \leq k < \ell \leq N,$$
$$M_{kk}(t) = x_k + B^{\mathrm{R}}_{kk}(t), \quad 1 \leq k \leq N,$$

where $B^{\mathrm{R}}_{k\ell}(t), B^{\mathrm{I}}_{k\ell}(t)$, $1 \leq k \leq \ell$ are independent Brownian motions and $(x_1, x_2, \cdots, x_N)$ is an element of

$$\mathbb{W}_N = \Big\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_N) : x_1 < x_2 < \cdots < x_N \Big\}.$$

The eigenvalue process of $M(t)$ solves the equation of Dyson's Brownian motion model [JMP 62]:

$$X_j(t) = x_j + B_j(t) + \sum_{\substack{k:1\le k\le N \\ k\neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \quad 1 \le j \le N \qquad (1)$$

and represents a system of N-noncolliding Brownian motions

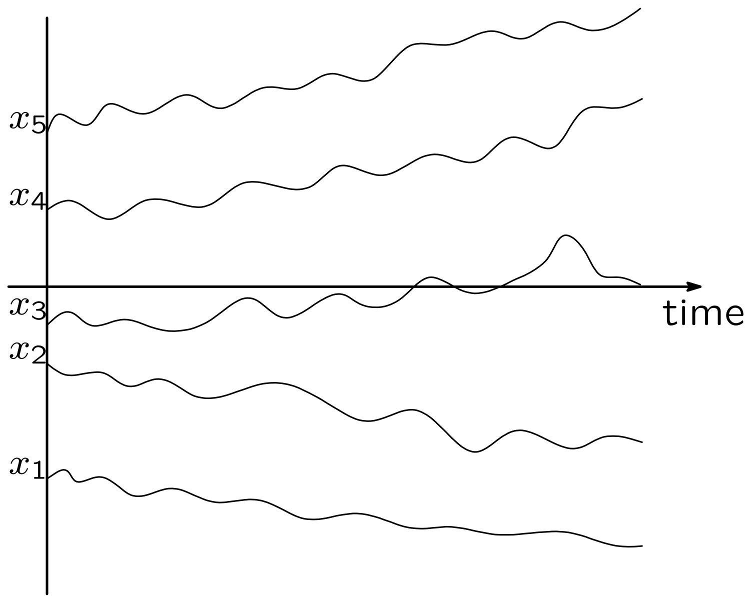

Figure 1: Noncolliding Brownian motions

A system of N-noncolliding Brownian motions is the $\mathbb{W}_N$-valued diffusion process with transition density

$$p_N(t-s, \boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} f_N(t-s, \boldsymbol{y}|\boldsymbol{x}), \quad 0 \leq s < t < \infty, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N,$$

where

$$f_N(t, \boldsymbol{y}|\boldsymbol{x}) = \det_{1 \leq j,k \leq N} \Big[p(t, y_j|x_k) \Big]$$

(**Karlin-McGregor determinant**)

$$h_N(\boldsymbol{x}) = \prod_{1 \leq j < k \leq N} (x_k - x_j) = \det_{1 \leq j,k \leq N} \Big[x_k^{j-1} \Big]$$

(**Vandermonde determinant**)

$$p(t, y|x) = \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}}, \quad t > 0, \; x \in \mathbb{R} \quad \text{(heat kernel)}$$

In this talk we discuss about the following:

1. Dyson's model is determinantal, that is its correlation functions are given by the determinants of some correlation kernel.

2. Under some conditions on an initial configuration, Dyson's model with an infinite number of particles can be consturcted as a limit of that with a finite number of particles.

3. The model has a stationary measure and the stationary process is Markovian.

The configuration space of unlabeled particles:

$$\mathfrak{M} = \Big\{\xi : \xi \text{ is a nonnegative integer valued Radon measures in } \mathbb{R}\Big\}.$$

Any element ξ of $\mathfrak{M}$ can be represented as:

$$\xi(\cdot) = \sum_{j\in\Lambda} \delta_{x_j}(\cdot)$$

with some sequence $(x_j)_{j\in\Lambda}$ of $\mathbb{R}$ satisfying $\sharp\{j \in \lambda : x_j \in K\} < \infty$, for any compact set K. Remind that $\Lambda = \mathbb{N}$ or is a finite subset of $\mathbb{N}$.
$\mathfrak{M}$ is a Polish space with the vague topology: we say ξ_n converges to ξ vaguely, if

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)\xi_n(dx) = \int_{\mathbb{R}} \varphi(x)\xi(dx)$$

for any $\varphi \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the set of all continuous real-valued functions with compact supports.

We introduce the following operations on $\mathfrak{M}$;

(restriction) for $A \subset \mathbb{R}$, $(\xi \cap A)(\cdot) = \sum_{j \in \Lambda : x_j \in A} \delta_{x_j}(\cdot)$,

(shift) for $u \in \mathbb{R}$, $\tau_u \xi(\cdot) = \sum_{j \in \Lambda} \delta_{x_j + u}(\cdot)$,

(dilatation) for $c \in \mathbb{R}$, $c \circ \xi(\cdot) = \sum_{j \in \Lambda} \delta_{c x_j}(\cdot)$.

(square) $\xi^{\langle 2 \rangle}(\cdot) = \sum_{j \in \Lambda} \delta_{x_j^2}(\cdot)$.

As an $\mathfrak{M}$-valued process $(\mathbb{P}, \Xi(t), t \in [0,\infty))$, we consider the system such that, for any integer $M \geq 1$, $f_m \in \mathrm{C}_0(\mathbb{R}), \theta_m \in \mathbb{R}, 1 \leq m \leq M$, $0 < t_1 < \cdots < t_M < \infty$,

$$\mathbb{E}\left[\exp\left\{\sum_{m=1}^{M} \theta_m \int_{\mathbb{R}} f_m(x)\Xi(t_m, dx)\right\}\right]$$

can be expanded with $\chi_m(x) = e^{\theta_m f_m(x)} - 1, 1 \leq m \leq M$ as

$$\begin{aligned} \mathcal{G}^{\xi}[\chi] \equiv & \sum_{N_1 \geq 0} \cdots \sum_{N_M \geq 0} \prod_{m=1}^{M} \frac{1}{N_m!} \int_{\mathbb{R}^{N_1}} \prod_{j=1}^{N_1} dx_j^{(1)} \cdots \int_{\mathbb{R}^{N_M}} \prod_{j=1}^{N_M} dx_j^{(M)} \\ & \times \prod_{m=1}^{M} \prod_{j=1}^{N_m} \chi_m\left(x_j^{(m)}\right) \rho\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right), \end{aligned}$$

where $\boldsymbol{x}_{N_m}^{(m)}$ denotes $(x_1^{(m)}, \ldots, x_{N_m}^{(m)}), 1 \leq m \leq M$.

In such a system $\rho(t_1, \boldsymbol{x}_{N_1}^{(1)}; \ldots; t_M, \boldsymbol{x}_{N_M}^{(M)})$ is called the $(N_1, \ldots, N_M)$-multitime correlation function and $\mathcal{G}^{\xi}[\chi]$ the generating function of multitime correlation functions.

1. Determinantal processes

Definition 1. An $\mathfrak{M}$-valued process $(\mathbb{P}, \Xi(t), t \in [0, \infty))$ is said to be determinantal with the correlation kernel $\mathbb{K}$, if the multitime correlation function is given by

$$\rho\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; t_2, \boldsymbol{x}_{N_2}^{(2)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right) = \det_{\substack{1 \leq j \leq N_m, 1 \leq k \leq N_n \\ 1 \leq m,n \leq M}} \left[\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\right],$$

for any $M \geq 1$,

any sequence $\{N_m\}_{m=1}^{M}$ of positive integers,

any time sequence $0 < t_1 < \cdots < t_M < \infty$.

$$(\boldsymbol{x}_{N_1}^{(1)}, \boldsymbol{x}_{N_2}^{(2)}, \dots, \boldsymbol{x}_{N_M}^{(M)}) \in \prod_{m=1}^{M} \mathbb{R}^{N_m}.$$

THEOREM 1. The non-colliding Brownian motions starting from the fixed configuration $\xi_N \in \mathfrak{M}$ with $\xi_N(\mathbb{R}) = N < \infty$ at time $t = 0$, is a determinantal process associated with the correlation kernel $\mathbb{K}^{\xi_N}$ given by

$$\mathbb{K}^{\xi_N}(s,x;t,y) = \frac{1}{2\pi i} \oint_{\Gamma(\xi_N)} dz\, p(s,x|z) \int_{\mathbb{R}} dw\, \frac{p(t,-iy|w)}{iw-z} \prod_{x' \in \xi_N} \frac{iw-x'}{z-x'}$$

$$-\mathbf{1}(s>t)p(s-t,x|y), \quad s,t \in [0,\infty),\ x,y \in \mathbb{R}, \tag{2}$$

where $\Gamma(\xi_N)$ is a closed contour on the complex plane $\mathbb{C}$ encircling the points in $\text{supp}\, \xi_N$ on the real line $\mathbb{R}$ once in the positive direction, and

$$\prod_{x \in \xi_N} f(x) = \prod_{x \in \text{supp}\xi_N} f(x)^{\xi(x)}.$$

When $\xi_N(x) \le 1$ for any $x \in \xi_N$, that is, there is no multiple point in the configuration ξ_N,

$$\mathbb{K}^{\xi_N}(s,x;t,y) = \int_{\mathbb{R}} \xi_N(dx')\, p(s,x|x') \int_{\mathbb{R}} dw\, p(t,-iy|w) \Phi(\xi_N, x', iw)$$

$$-\mathbf{1}(s>t) p(s-t, x|y).$$

where for $a \in \xi_N$ and $z \in \mathbb{C}$

$$\Phi(\xi_N, a, z) = \prod_{x \in \mathrm{supp}\xi_N \setminus \{a\}} \frac{x-z}{x-a} = \prod_{x \in \mathrm{supp}\xi_N \setminus \{a\}} \left(1 - \frac{z-a}{x-a}\right).$$

$\Phi(\xi_N, a, z)$ is an entire function whose zero set is supp $\xi_N \setminus \{a\}$.

Multiple Hermite polynomials

(**Ref**: Bleher and Kuijlaars, Ann. Inst. Fourier, (2005))

We introduce the following functions:

$$P_\xi(y) = \int_{\mathbb{R}} dw \, \frac{e^{-(w+iy)^2/2}}{\sqrt{2\pi}} \prod_{x \in \xi} (iw - x),$$

$$Q_\xi(y) = \frac{1}{2\pi i} \oint_{\Gamma(\xi)} dz \, \frac{e^{-(z-y)^2/2}}{\sqrt{2\pi}} \frac{1}{\prod_{x \in \xi}(z - x)}.$$

Then we define for $\xi^N = \sum_{k=1}^N \delta_{x_k}$ with $x_1 \le x_2 \le \cdots \le x_N$

$$H_j^{(-)}(y; \xi^N) = P_{\xi_j}(y), \quad H_j^{(+)}(y; \xi^N) = Q_{\xi_{j+1}}(y), \quad 0 \le j \le N-1,$$

where we put $\xi_j = \sum_{k=1}^j \delta_{x_k}$.

The following relations hold:

$$\int_{\mathbb{R}} dy\, H_j^{(-)}(y;\xi^N) H_k^{(+)}(y;\xi^N) = \delta_{jk}, \quad 0 \le j,k \le N-1.$$

$$\int_{\mathbb{R}} dy\, H_j^{(-)}\left(\frac{y}{\sqrt{t}}; \frac{1}{\sqrt{t}} \circ \xi^N\right) p(t-s,y|x) = \left(\frac{s}{t}\right)^{j/2} H_j^{(-)}\left(\frac{x}{\sqrt{s}}; \frac{1}{\sqrt{s}} \circ \xi^N\right),$$

$$\int_{\mathbb{R}} dx\, p(t-s,y|x) H_j^{(+)}\left(\frac{x}{\sqrt{s}}; \frac{1}{\sqrt{s}} \circ \xi^N\right) = \left(\frac{s}{t}\right)^{(j+1)/2} H_j^{(+)}\left(\frac{y}{\sqrt{t}}; \frac{1}{\sqrt{t}} \circ \xi^N\right).$$

Then

$$\mathbb{K}^{\xi_N}(s,x;t,y) = \frac{1}{\sqrt{s}} \sum_{j=0}^{N-1} \left(\frac{t}{s}\right)^{j/2} H_j^{(+)}\left(\frac{x}{\sqrt{s}}; \frac{1}{\sqrt{s}} \circ \xi^N\right) H_j^{(-)}\left(\frac{y}{\sqrt{t}}; \frac{1}{\sqrt{t}} \circ \xi^N\right)$$
$$-\mathbf{1}(s>t) p(s-t,x|y).$$

2. Dyson's model with an infinite number of particles

Spohn[87] considered an infinite particle system obtained by taking the $N \to \infty$ limit of (1) and called the system Dyson's model. He studied the equilibrium dynamics with respect to the determinantal (Fermion) point process μ_{sin}, in which any spatial correlation function ρ_m is given by a determinant with the sine kernel:

$$K_{\text{sin}}(y-x) = \frac{\sin\{\pi(y-x)\}}{\pi(y-x)}, \quad x, y \in \mathbb{R}. \tag{3}$$

Osada[CMP 96] constructed the infinite particle system represented by a diffusion process, which has μ_{sin} as a reversible measure, by the Dirichlet form approach. Recently he proved that this system satisfies the SDEs (1) with $N = \infty$[Preprint 09].

Eynard and Mehta[J.Phys A 98] showed that multi-time correlation functions for the process (1) are generally given by determinants, if the process starts from $\mu_{N,\sigma^2}^{\mathrm{GUE}}$, eigenvalue distribution of GUE with variance σ^2.

Nagao and Forrester[Phys. Lett. A 98] evaluated the bulk scaling limit $N = \sigma^2 \to \infty$ and derived the so-called extended sine kernel:

$$\mathbf{K}_{\sin}(t-s, y-x) = \begin{cases} \displaystyle\int_0^1 du\, e^{\pi^2 u^2 (t-s)/2} \cos\{\pi u(y-x)\} & \text{if } t > s \\ K_{\sin}(y-x) & \text{if } t = s \\ \displaystyle -\int_1^\infty du\, e^{\pi^2 u^2 (t-s)/2} \cos\{\pi u(y-x)\} & \text{if } t < s, \end{cases} \quad (4)$$

$s, t \geq 0, x, y \in \mathbb{R}$.

Since $\lim_{N\to\infty} \mu_{N,N}^{\mathsf{GUE}} = \mu_{\mathsf{sin}}$, the process, whose multitime correlation functions are given by determinant with the extended sine kernel, is expected to be identified with the infinite-dimensional equilibrium dynamics of Spohn and Osada. This equivalence is, however, not yet been proved. In fact the Markov property of the former process was not proved.

Fritz [AOP 87] established the theory of non-equilibrium dynamics of one-dimensional infinite particle systems with a finite-range hard-core potential.

We study the non-equilibrium dynamics of Dyson's model, which is an infinite particle system with a long-range log-potential.

For $L > 0, \alpha > 0$ and $\xi \in \mathfrak{M}$ we put

$$M(\xi, L) = \int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{x}, \qquad M_\alpha(\xi, L) = \left(\int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{|x|^\alpha}\right)^{1/\alpha}$$

and

$$M(\xi) = \lim_{L\to\infty} M(\xi, L), \qquad M_\alpha(\xi) = \lim_{L\to\infty} M_\alpha(\xi, L),$$

if the limits exist. We introduce the following two conditions:

(**C.1**) there exists $C_0 > 0$ such that $|M(\xi)| \le C_0$,

(**C.2**) there exist $\alpha \in (1, 2)$ and $C_1 > 0$ such that $M_\alpha(\xi) \le C_1$,
(ii) there exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a^2}\xi^{\langle 2\rangle}) \le C_2(|a| \vee 1)^{-\beta} \quad \forall a \in \mathrm{supp}\xi.$$

The configuration spaced $\mathfrak{X}$ and $\mathfrak{X}_0$.

We put

$$\mathfrak{X} = \left\{ \xi \in \mathfrak{M} : \ \xi \text{ satisfies the conditions } (\mathbf{C.1}) \text{ and } (\mathbf{C.2}) \right\},$$

and $\mathfrak{X}_0 = \{\xi \in \mathfrak{X} : \xi(\{x\}) \leq 1, \ x \in \mathbb{R}\}$.

For $\xi \in \mathfrak{X}_0$, $a \in \mathbb{R}$ and $z \in \mathbb{C}$ we put

$$\Phi(\xi, a, z) = \lim_{L \to \infty} \Phi(\xi \cap [a - L, a + L], a, z),$$

We note that

$$|M(\tau_{-a}\xi)| < \infty, \ |M_2(\tau_{-a}\xi)| < \infty \Longrightarrow \Phi(\xi, a, z) \in (0, \infty),$$

and $\Phi(\xi, a, z)$ is an entire function whose zero set is supp $\xi \setminus \{a\}$.

THEOREM 2. Let $\xi \in \mathfrak{X}_0$. Then

$$(\mathbb{P}_{\xi \cap [-N,N]}, \Xi(t), t \in [0,\infty)) \longrightarrow (\mathbb{P}_{\xi}, \Xi(t), t \in [0,\infty)), \quad N \to \infty,$$

weakly on the space $C([0,\infty) \to \mathfrak{M})$ of $\mathfrak{M}$-valued continuous functions, and the limit process $(\mathbb{P}_{\xi}, \Xi(t), t \in [0,\infty))$ is determinantal with the correlation kernel

$$\mathbb{K}^{\xi}(s,x;t,y) = \int_{\mathbb{R}} \xi(dx')\, p(s,x|x') \int_{\mathbb{R}} dw\, p(t,-iy|w) \Phi(\xi,x',iw)$$

$$-\mathbf{1}(s>t)p(s-t,x|y).$$

Examples. For $\kappa > 0$, we put

$$g^{\kappa}(x) = \mathrm{sgn}(x)|x|^{\kappa}, \ x \in \mathbb{R}, \ \text{and } \eta^{\kappa} = \sum_{\ell \in \mathbb{Z}} \delta_{g^{\kappa}(\ell)}$$

η^{κ} is an element of $\mathfrak{X}_0$, in case $\kappa > 1/2$.

We denote by $(\mathbf{P}_{\mathsf{sin}}, \Xi(t), t \in [0,\infty))$ is the process which is determinantal with the extended sine kernel $\mathbf{K}_{\mathsf{sin}}$. The process is a reversible process with reversible probability measure μ_{sin} which is determinantal with the sine kernel $K_{\mathsf{sin}}(x,y)$.

For the measure $\xi^{\mathbb{Z}}(\cdot) = \eta^1(\cdot) = \sum_{\ell\in\mathbb{Z}} \delta_\ell(\cdot)$ we can show that

$$\textbf{Long-time limit}: \lim_{u\to\infty} \mathbb{K}^{\xi^{\mathbb{Z}}}(u+s,x;u+t,y) = \mathbf{K}_{\mathsf{sin}}(s,x;t,y),$$

which implies that

$$(\mathbb{P}_{\xi^{\mathbb{Z}}}, \Xi(t), t \in [0,\infty)) \longrightarrow (\mathbf{P}_{\mathsf{sin}}, \Xi(t), t \in [0,\infty)),$$

weakly in the sense of finite dimensional distributions.

The configuration space $\mathfrak{Y}$.

We introduce another condition:

(C.3) there exists $\kappa \in (1/2, 1)$ and $m \in \mathbb{N}$ such that

$$m(\xi, \kappa) \equiv \max_{k \in \mathbb{Z}} \xi\Big([g^{\kappa}(k), g^{\kappa}(k+1)] \Big) \le m.$$

We denote by $\mathfrak{Y}_m^{\kappa}$ the set of configuration of ξ satisfying **(C.1)** and **(C.3)** with κ, m, and put

$$\mathfrak{Y} = \bigcup_{\kappa \in (1/2,1)} \bigcup_{m \in \mathbb{N}} \mathfrak{Y}_m^{\kappa}$$

Noting that the set $\{\xi \in \mathfrak{M} : m(\xi, \kappa) \le m\}$ is relatively compact for each $\kappa > 0$ and $m \in \mathbb{N}$, we see that $\mathfrak{Y}$ is locally compact.

Definition 2. We call ξ_n converges to ξ Φ-moderately if

$$\lim_{n \to \infty} \Phi(\xi_n, \sqrt{-1}, \cdot) = \Phi(\xi, \sqrt{-1}, \cdot)$$

uniformly on any compact set of $\mathbb{C}$.

It is easy to see that ξ_n converges to ξ Φ-moderately if ξ_n converges to ξ vaguely and

$$\lim_{L \to \infty} \sup_{n > 0} \left| \int_{[-L,L]^c} \frac{\xi^n(dx)}{x} \right| = 0,$$

$$\lim_{L \to \infty} \sup_{n > 0} \int_{[-L,L]^c} \frac{\xi^n(dx)}{x^2} = 0.$$

THEOREM 3. (1) Suppose that $\xi \in \mathfrak{Y}$. Then

$$(\mathbb{P}_{\xi \cap [-N,N]}, \Xi(t), t \in [0,\infty)) \longrightarrow (\mathbb{P}_{\xi}, \Xi(t), t \in [0,\infty)), \quad N \to \infty,$$

weakly on the space $C([0,\infty) \to \mathfrak{M})$, and the limit process $(\mathbb{P}_{\xi}, \Xi(t), t \in [0,\infty))$ is determinantal with the correlation kernel $\mathbb{K}^{\xi}(s,x;t,y)$.

(2) Suppose that $\xi, \xi^n \in \mathfrak{Y}_m^{\kappa}, n \in \mathbb{N}$, for some $\kappa \in (1/2, 1)$ and $m \in \mathbb{N}$. If ξ_n converges to ξ Φ-moderately, then

$$(\mathbb{P}_{\xi^n}, \Xi(t), t \in [0,\infty)) \longrightarrow (\mathbb{P}_{\xi}, \Xi(t), t \in [0,\infty)), \quad N \to \infty,$$

weakly on the space $C([0,\infty) \to \mathfrak{M})$.

3. Markov property

For $\xi \in \mathfrak{Y}$ and a bounded continuous function on $\mathfrak{M}$ we put

$$T_t f(\xi) = \mathbb{E}_\xi[f(\Xi(t))].$$

THEOREM 4 (Markov property).

(1) $\mu_{\text{sin}}(\mathfrak{Y}) = 1$ and T_t is extended to the operator on $L^2(\mathfrak{M}, \mu_{\text{sin}})$, and $\mathbf{P}_{\text{sin}}(\cdot) = \int_{\mathfrak{M}} \mu_{\text{sin}}(d\xi)\mathbb{P}_\xi(\cdot)$.

(2) The reversible process $(\mathbf{P}_{\text{sin}}, \Xi(t), t \in [0, \infty))$ is a Markov process, that is,

$$\begin{aligned} & \mathbf{E}_{\text{sin}}\Big[f_0(\Xi(0))f_1(\Xi(t_1))f_2(\Xi(t_2))\Big] \\ = \; & \mathbf{E}_{\text{sin}}\Big[f_0(\Xi(0))f_1(\Xi(t_1))T_{t_2-t_1}f_2(\Xi(t_1))\Big] \\ = \; & \int_{\mathfrak{M}} \mu_{\text{sin}}(d\xi) f_0(\xi) T_{t_1}(f_1 T_{t_2-t_1} f_2)(\xi) \end{aligned} \tag{5}$$

for any $0 \le t_1 < t_2 < \infty$ and any bounded continuous functions f_0, f_1 and f_2.

Let $\mathfrak{S} \subset \mathfrak{M}$ and

$C(\mathfrak{S})$: the set of all vaguely continuous functons on $\mathfrak{S}$.

$\widehat{C}(\mathfrak{S})$: the set of all Φ-moderately continuous functions on $\mathfrak{S}$.

THEOREM 3 (2) implies that for any $f \in C(\mathfrak{M})$ and $0 < s < t$

$\mathbb{E}_{\xi^n}[f(\Xi(s))g((\Xi(t))] \longrightarrow \mathbb{E}_{\xi^n}[f(\Xi(s))g((\Xi(t))], \quad \xi^n \to \xi, \Phi$-moderately.

THEOREM 4 is derived by finding a configuration space $\widehat{\mathfrak{Y}} \subset \mathfrak{Y}$ such that $\mu_{\text{sin}}(\widehat{\mathfrak{Y}}) = 1$ and $C(\widehat{\mathfrak{Y}}) = \widehat{C}(\widehat{\mathfrak{Y}})$.

Conjecture. There exists a configuration space $\widehat{\mathfrak{Y}}$ such that

$$T_t f \in \widehat{C}(\widehat{\mathfrak{Y}}), \qquad \text{for } f \in \widehat{C}(\widehat{\mathfrak{Y}}).$$