

Convergence of the weighted quadratic variations of some fractional Brownian sheets

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Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a regular deterministic function, $W := (W_{(s,t)})_{(s,t) \in [0,1]^2}$ a two-parameter stochastic process and $G_n := \{(\frac{i}{n}, \frac{j}{n}), 1 \leq i, j \leq n\}$ a regular grid with mesh $1/n$. In this talk we study the asymptotic behavior as n tends to infinity of the following quantity

$$\sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f\left(W_{(\frac{i-1}{n}, \frac{j-1}{n})}\right) |\Delta_{i,j} W|^2 \quad n \geq 1, (s, t) \in [0, 1]^2$$

where

$$\Delta_{i,j} W := W_{(\frac{i-1}{n}, \frac{j-1}{n})} + W_{(\frac{i}{n}, \frac{j}{n})} - W_{(\frac{i-1}{n}, \frac{j}{n})} - W_{(\frac{i}{n}, \frac{j-1}{n})}.$$

We are interested in two situations:

- W is a **standard** Brownian sheet,
- W is a **fractional** Brownian sheet.

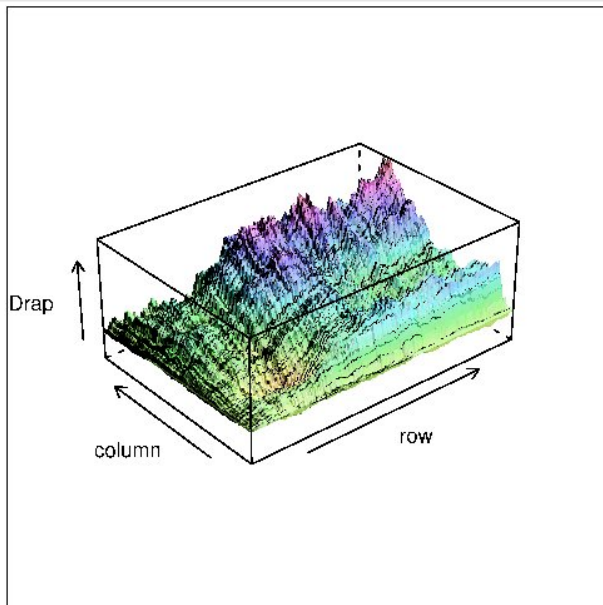
Let $W = (W_{(s,t)})_{(s,t) \in [0,1]^2}$ be a two-parameter Brownian motion that is,

- $W_{(s,t)} = 0$, if $s = 0$ or $t = 0$,
- W is a centered Gaussian process with covariance function

$$\mathbb{E} [W_{(s,t)} W_{(s',t')}] = (s \wedge s')(t \wedge t'), \quad (s, t), (s', t') \in [0, 1]^2.$$

Remark that $\Delta_{i,j} W$ and $\Delta_{i',j'} W$ are two independent rv if $(i, j) \neq (i', j')$.

The standard Brownian sheet



Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a regular enough deterministic function. Let $X_n = (X_n(s, t))_{(s,t) \in [0,1]^2}$ defined as,

$$X_n(s, t) := n \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f \left(W_{\left(\frac{i-1}{n}, \frac{j-1}{n}\right)} \right) \left(|\Delta_{i,j} W|^2 - \frac{1}{n^2} \right), \quad n \geq 1.$$

Theorem (R.)

Under some regularity assumption on f we have that

$$X_n(\cdot, \bullet) := n \sum_{i=1}^{[n\cdot]} \sum_{j=1}^{[n\bullet]} f \left(W_{\left(\frac{i-1}{n}, \frac{j-1}{n}\right)} \right) \left(|\Delta_{i,j} W|^2 - \frac{1}{n^2} \right) \xrightarrow[n \rightarrow \infty]{\text{law}(S)} X,$$

in the Skorohod space $\mathcal{D}([0, 1]^2)$ and X is a non-Gaussian continuous process defined as

$$X_{(s,t)} := \sqrt{2} \int_{[0,s] \times [0,t]} f(W_\rho) dB_\rho.$$

The previous theorem has been established for the standard Brownian motion by Jacod, Gradinaru and Nourdin, Aït-Sahalia and Jacod, ...

Let

$$Y_{(s,t)} = \int_{[0,s] \times [0,t]} \sigma(W_\rho) dW_\rho + \int_{[0,s] \times [0,t]} M_\rho d\rho,$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ and M is a continuous two-parameter process.

We observe an unique path of Y on $\left\{ \left(\frac{i}{n}, \frac{j}{n} \right), 1 \leq i, j, \leq n \right\}$ with $n \geq 1$. Set

$$\left\{ \begin{array}{l} V_{(s,t)}^n := \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_{i,j} Y|^2, \quad (s, t) \in [0, 1]^2, \quad n \geq 1, \\ C_{(s,t)} := \int_{[0,s] \times [0,t]} \sigma^2(W_{(u,v)}) \, dudv, \quad (s, t) \in [0, 1]^2, \\ Y_{(s,t)}^n := n(V_{(s,t)}^n - C_{(s,t)}), \quad (s, t) \in [0, 1]^2. \end{array} \right.$$

Let

$$V_n(s, t) := \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} |\Delta_{i,j} Y|^2, \quad (s, t) \in [0, 1]^2.$$

Proposition

The estimator V_n of C is consistent:

$$V_n(s, t) \xrightarrow[n \rightarrow \infty]{L^2} C(s, t), \quad (s, t) \in [0, 1]^2.$$

Rate of convergence this consistent estimator?

Proposition

Under some regularity assumptions on $\sigma(\cdot)$, the sequence on two-parameter processes

$$Y_n(\cdot, \bullet) := n (V_n - C)(\cdot, \bullet) \xrightarrow[n \rightarrow \infty]{law(S)} \sqrt{2} \int_{[0, \cdot] \times [0, \bullet]} \sigma^2(W_\rho) dB_\rho$$

in the Skorohod space $D([0, 1]^2)$.

Theorem

Let,

$$S_n(s, t) := n^2 \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_{i,j} Y|^4, \quad S(s, t) := 3 \int_{[0,s] \times [0,t]} \sigma^4(W_\rho) d\rho.$$

We have for every (s, t) in $(0, 1]^2$ that

$$S_n(s, t)^{-1/2} Y_n(s, t) \xrightarrow[n \rightarrow \infty]{law(S)} \sqrt{\frac{2}{3}} N, \quad N \sim \mathcal{N}(0, 1).$$

Lemme (Aldous and Eagleson)

If $(Y_n)_n$ converges stably in law to Y and if $(S_n)_n$ converges in probability to S then,

$$(Y_n, S_n)_n \xrightarrow{law} (Y, S).$$

Let $B^H := (B_t^H)_{t \in [0,1]}$ be a fractional Brownian motion with Hurst index $0 < H < 1$.

Given a deterministic function $f : \mathbb{R} \rightarrow \mathbb{R}$ regular enough we are interested in the asymptotic behavior of the following quantity as n tends to infinity,

$$V_n(f) := \sum_{k=0}^{n-1} f(B_{k/n}^H) \left[n^{2H} (B_{\frac{k+1}{n}}^H - B_{\frac{k}{n}}^H)^2 - 1 \right]$$

The non-weighted case ($f(\cdot) = 1$) has been investigated by Breuer and Major, Dobrushin and Major, Giraitis and Surgailis, Taqqu who have shown that the value $H = 3/4$ is a particular value in this study.

- If $H \in (0, 1/4)$ then

$$n^{2H-1} V_n(f) \xrightarrow[n \rightarrow \infty]{L^2} \frac{1}{4} \int_0^1 f''(B_s^H) ds,$$

- if $1/4 < H < 3/4$ then

$$n^{-1/2} V_n(f) \xrightarrow[n \rightarrow \infty]{law(S)} C_H \int_0^1 f(B_s^H) dW_s,$$

W standard Brownian motion independent of B^H ,

- if $H = 3/4$ then

$$(n \log(n))^{-1/2} V_n(f) \xrightarrow[n \rightarrow \infty]{law(S)} C_{3/4} \int_0^1 f(B_s^{3/4}) dW_s,$$

W standard Brownian motion independent of $B^{1/4}$,

- if $H > 3/4$ then

$$n^{1-2H} V_n(f) \xrightarrow[n \rightarrow \infty]{L^2} \int_0^1 f(B_s^H) dZ_s, \text{ where } Z \text{ is a Rosenblatt process.}$$

The case $H = 1/4$ is not contained in the previous result.

Theorem (Nourdin, R.)

$$n^{-1/2} V_n(f) = n^{-1/2} \sum_{k=0}^{n-1} f(B_{k/n}^{1/4}) \left[\sqrt{n} (B_{\frac{k+1}{n}}^{1/4} - B_{\frac{k}{n}}^{1/4})^2 - 1 \right]$$

$$\xrightarrow[n \rightarrow \infty]{\text{law}(S)} C_{1/4} \int_0^1 f(B_s^{1/4}) dW_s + \frac{1}{4} \int_0^1 f''(B_s^{1/4}) ds,$$

W standard Brownian motion independent of B^H .

For $f : \mathbb{R} \rightarrow \mathbb{R}$ regular enough we aim at defining

$$\int_0^1 f(B_s^{1/4}) \circ dB_s^{1/4} := \lim_{n \rightarrow \infty} \begin{cases} S_n(f) \\ T_n(f) \end{cases}$$

where

$$S_n(f) = \sum_{k=0}^{n-1} \frac{f(B_{k/n}^{1/4}) + f(B_{(k+1)/n}^{1/4})}{2} (B_{(k+1)/n}^{1/4} - B_{k/n}^{1/4}),$$

$$T_n(f) = \sum_{k=1}^{\lfloor n/2 \rfloor} f(B_{(2k-1)/n}^{1/4}) (B_{(2k)/n}^{1/4} - B_{(2k-2)/n}^{1/4}).$$

Theorem (Gradinaru, Nourdin, Russo, Vallois; Cheridito, Nualart)

It holds that

$$\int_0^1 f'(B_s^{1/4}) d^\circ B_s^{1/4} := \lim_{n \rightarrow \infty} S_n(f') \quad \text{exists in probability}$$

with

$$\int_0^1 f'(B_s^{1/4}) d^\circ B_s^{1/4} = f(B_1^{1/4}) - f(0).$$

Theorem (Nourdin, R.; Burdzy, Swanson)

$$\int_0^1 f'(B_s^{1/4}) d^* B_s^{1/4} := \lim_{n \rightarrow \infty} T_n(f') \quad \text{exists in law}$$

and satisfy

$$\int_0^1 f'(B_s^{1/4}) d^* B_s^{1/4} \stackrel{\text{law}}{=} f(B_1^{1/4}) - f(0) - \frac{\kappa}{2} \int_0^1 f''(B_s^{1/4}) dW_s$$

where κ denotes an explicit constant and W is a standard BM independent of $B^{1/4}$.

Theorem (Nourdin, R., Swanson)

$$\int_0^1 f'(B_s^{1/6}) d^\circ B_s^{1/6} := \lim_{n \rightarrow \infty} S_n(f') \quad \text{exists in law}$$

and satisfy

$$\int_0^1 f'(B_s^{1/6}) d^\circ B_s^{1/6} \stackrel{\text{law}}{=} f(B_1^{1/6}) - f(0) - \frac{\kappa_{1/6}}{2} \int_0^1 f'''(B_s^{1/6}) dW_s$$

where $\kappa_{1/6}$ denotes an explicit constant and W is a standard BM independent of $B^{1/6}$.

Definition (Ayache, Léger and Pontier)

A fractional Brownian sheet $(B_{(s,t)}^{\alpha,\beta})_{(s,t) \in [0,1]^2}$ with Hurst indices $(\alpha, \beta) \in (0, 1)^2$ is a centered two-parameter Gaussian process equal with

$$\{(s, t) \in [0, 1]^2, s = 0 \text{ or } t = 0\}$$

whose covariance function is given by,

$$\begin{aligned} R^{\alpha,\beta}((s_1, t_1), (s_2, t_2)) &:= \mathbb{E} \left[B_{(s_1, t_1)}^{\alpha,\beta} B_{(s_2, t_2)}^{\alpha,\beta} \right] \\ &= K^\alpha(s_1, s_2) K^\beta(t_1, t_2) \\ &= \frac{1}{2} (s_1^{2\alpha} + s_2^{2\alpha} - |s_1 - s_2|^{2\alpha}) \frac{1}{2} (t_1^{2\beta} + t_2^{2\beta} - |t_1 - t_2|^{2\beta}). \end{aligned}$$

$(B_{(s,t)}^{\alpha,\beta})_{(s,t) \in [0,1]^2}$ is a semimartingale if and only if $(\alpha, \beta) = (1/2, 1/2)$ and in this case this is the standard Brownian sheet studied previously.

Theorem (R.)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a deterministic function regular enough. Let

$$X_n(s, t) := n^{-1} \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} f \left(B_{\left(\frac{i-1}{n}, \frac{j-1}{n}\right)}^{\alpha, \beta} \right) \left(n^{2(\alpha+\beta)} |\Delta_{i,j} B^{\alpha, \beta}|^2 - 1 \right), \quad (s, t) \in [0, 1]^2$$

be the re-normalized weighted quadratic variations where the increments $\Delta_{i,j} B^{\alpha, \beta}$ are defined as

$$\Delta_{i,j} B^{\alpha, \beta} := B_{\left(\frac{i-1}{n}, \frac{j-1}{n}\right)}^{\alpha, \beta} + B_{\left(\frac{i}{n}, \frac{j}{n}\right)}^{\alpha, \beta} - B_{\left(\frac{i-1}{n}, \frac{j}{n}\right)}^{\alpha, \beta} - B_{\left(\frac{i}{n}, \frac{j-1}{n}\right)}^{\alpha, \beta}.$$

If $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$ with $\alpha + \beta > \frac{1}{2}$ then

$$X_n \xrightarrow[n \rightarrow \infty]{\text{fdd-law}(S)} X$$

with $X_{s,t} := \sigma_{\alpha, \beta} \int_{[0,s] \times [0,t]} f \left(B_{(u,v)}^{\alpha, \beta} \right) dW_{(u,v)}$, $(s, t) \in [0, 1]^2$ where W is a standard Brownian sheet independent of $B^{\alpha, \beta}$.