Large Deviation Multifractal Analysis of a Class of Additive Processes with Correlated Non-Stationary Increments

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1 The model

Let $\{\lambda_i\}_i$ be a nondecreasing sequence of positive real numbers. Given i, let $\{\tau_j^{(i)}\}_j$ be a sequence of independent exponential random variables with parameter λ_i. Set

$$T_k^{(i)} = \sum_{j=0}^{k} \tau_j^{(i)}$$

Let $Z_i(t)$ be a random process satisfying

$$Z_i(0) = 0,$$

$$\frac{d}{dt} Z_i(t) = 1,$$

$$Z_i((T_k^{(i)})_+) = \frac{1}{\mu} Z_i((T_k^{(i)})_-).$$

Let Z_i be all independent and define

$$Z = \sum_i Z_i$$

Z is almost surely everywhere finite iff

$$\sum \frac{1}{\lambda_i} < \infty. \tag{1}$$

We denote

$$\beta = \inf \left\{ \gamma \geq 1; \sum_i \lambda_i^{1-\gamma} < \infty \right\}.$$

Because of (1), $\beta \in [1, 2]$.

We fix $L > 1$. We denote $M_k = \sharp\{\lambda_i < L^k\}$ and $N_k = M_k - M_{k-1}$. We have

$$N_k < K_1 L^{k(\beta-1+\varepsilon_0)}$$

and (for some sequence $\{a_k\}$)

$$N_{a_k} > K_2 L^{a_k(\beta-1-\varepsilon_0)}.$$

The sequence $\{\lambda_i\}$ is *regular* if for any $\varepsilon_0 > 0$ the sequence $\{a_{k+1} - a_k\}$ is bounded.

Question: what is the stationary distribution of Z_i?

2 The question

The large deviation multifractal spectrum of Z is defined as

$$f(\alpha) = \lim_{\varepsilon \to 0} \limsup_{h \to 0} \frac{\log N_h^\varepsilon(\alpha)}{-\log h},$$

where $N_h^\varepsilon(\alpha)$ stands for

$$\sharp \left\{ k; \alpha - \varepsilon < \frac{\log |Z(t_0 + (k+1)h) - Z(t_0 + kh)|}{\log h} < \alpha + \varepsilon \right\}$$

For the upper bound it is enough to estimate the probability that $|Z(t+h) - Z(t)| \approx h^\alpha$. For the lower bound it is not enough (the increments are correlated).

It is easier to work with smaller α, for big α the estimations are very delicate.

3 The answer

Theorem 1. *For regular* $\{\lambda_i\}$ *we almost surely have*

$$f(\alpha) = \begin{cases} \beta\alpha & \text{if } \alpha \in [0, 1/\beta]; \\ 1 + 1/\beta - \alpha & \text{if } \alpha \in [1/\beta, 1 + 1/\beta]; \\ -\infty & \text{otherwise.} \end{cases}$$

Exception: for $\beta = 2$ *and* $\alpha > 1/2$ *we can only give the upper bound. Otherwise, given any* $1 \leq \beta_1 \leq \beta_2 \leq 2$ *there exits a sequence* $\{\lambda_i\}_i$ *such that almost surely:*

$$f(\alpha) = \begin{cases} \beta_2\alpha & \text{if } \alpha \in [0, 1/\beta_2]; \\ 1 & \text{if } \alpha \in [1/\beta_2, 1/\beta_1]; \\ 1 + 1/\beta_1 - \alpha & \text{if } \alpha \in [1/\beta_1, 1 + 1/\beta_1]; \\ -\infty & \text{otherwise.} \end{cases}$$

4 Elements of proof

4.1 $\alpha < 1/\beta$, upper bound

Set $\varepsilon_0 < (1 - \alpha\beta)/2$. We estimate the tail

$$T = \sum_{\lambda_i > h^{-\alpha}} (Z_i(t+h) - Z_i(t))$$

obtaining

$$P(|T| < h^\alpha/2) \geq 1 - c(\varepsilon_0) h^{1-\alpha\beta-\varepsilon_0}.$$

Let us denote this event by A. As

$$\sharp\{\lambda_i < h^{-\alpha}\} \leq K_2(\varepsilon_0) h^{-\alpha(\beta-1+\varepsilon_0)} << h^{\alpha-1},$$

the increment $Z(t+h) - Z(t)$ cannot be greater than h^α if A holds. We need only to estimate the probability that $Z(t+h) - Z(t) < -h^\alpha$. For this to happen we would need at least one of the elementary processes Z_i with $\lambda_1 < h^{-\alpha}$ to have a jump, probability of which is not greater than

$$\sum_{k < -\alpha \log h / \log L} N_k \left(1 - e^{hL^k}\right) \leq \sum_{k < -\alpha \log h / \log L} K_1(\varepsilon_0) h L^{k(\beta+\varepsilon_0)}$$

$$\approx c(\varepsilon_0) h^{1-\alpha\beta-\alpha\varepsilon_0}.$$

4.2 $\alpha > 1/\beta$, $\beta \in (1,2)$, lower bound

We choose small ε_0.

Proposition 1. *There exist constants K_3, K_4 and a sequence $m_i \to \infty$ such that*

$$N_{m_i} \geq K_3 M_{m_i}$$

$$\sum_{j > m_i} \frac{N_j}{L^j} \leq K_4 \frac{N_{m_i}}{L^{m_i}}$$

$$\limsup_{i \to \infty} \frac{\log N_{m_i}}{m_i \log L} \geq \beta - 1.$$

Choose one of those m_i and let

$$h = \frac{1}{L^m N_m}.$$

Denote

$$A_1(k) = \{ \sum_{\lambda_i > L^m} (Z_i(t_0 + (k+1)h) - Z_i(t_0 + kh))$$
$$\in (-L^{-m}/2, L^{-m} \log|\log h|)\}$$

We can estimate that the number of $0 \leq k \leq h^{-1} - 1$ for which $A_1(k)$ holds is with probability $1 - c/\log m$ not smaller than $h^{-1+\varepsilon_0}$.

The event $A_2(k)$ holds if all the elementary processes Z_i with $\lambda_i < L^{m-1}$ have no jump between $t_0 + kh$ and $t_0 + |(k+1)h$. This event holds with probability not smaller than some constant.

We put inside $[t_0, t_0 + 1]$ some intervals J_j, each of length $h^{\varepsilon_0} L^{-m}$ and in distance at least $h^{-\varepsilon_0} L^{-m}$ from each other. The event $A_3(k)$ holds if precisely one Z_i for $L^{m-1} < \lambda_i < L^m$ has a jump between $t_0 + kh$ and $t_0 + (k+1)h$ and if this Z_i had no jump before in the interval J_j to which $t_0 + kh$ belongs. The probability of this event is bounded from below by a constant.

We can thus choose roughly $h^{-1+3\varepsilon_0}$ small intervals $(t_0+k_ih, t_0+(k_i+1)h)$. At each of them the event $B(k_i)$ that the increment of Z is approximately $\pm h^\alpha$ happens with probability at least $h^{\alpha+\varepsilon_0-1/(\beta-\varepsilon_0)}$ (from estimations on the stationary distribution for elementary processes) independently of the previous results. More precisely, the events $B(k_i)$ are dependent but the conditional probability that $B(k_i)$ happens given any possible sequence of results for $B(k_1), \dots, B(k_{i-1})$ is bounded from below by the number above. We complete the proof using law of large numbers.