Law of the absorption time of positive self-similar Markov processes

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Let $X = ((X_t)_{t\geq 0}, (\mathbb{Q}_x)_{x>0})$ be a positive self-similar strong Markov process of index $\alpha > 0$.

Lamperti [72] states the following zero-one law: if

$$T_0 = \inf\{s > 0; X_s = 0\}$$

then, for all $x > 0$, either $\mathbb{Q}_x(T_0 < \infty) = 1$ or $\mathbb{Q}_x(T_0 < \infty) = 0$.

Our purpose is to describe the distribution function of T_0 in the former case.

The Lamperti mapping

Lamperti proved that for each fixed $\alpha > 0$, there is a bijective correspondence between $[0,\infty)$-valued self-similar Markov processes with index α and real-valued Lévy processes, i.e. processes with stationary and independent increments.
Let $((\xi_t)_{t\geq 0}, \mathbb{P})$ be a Lévy process and introduce the additive functional

$$A_t = \inf\{s > 0;\ \Sigma_s = \int_0^s e^{\alpha \xi_r} dr > t\}, \quad t \geq 0.$$

Lamperti showed that

$$X_t = x \exp\left(\xi_{A_{tx^{-\alpha}}}\right), \quad t \geq 0. \tag{1}$$

Let Ψ be the characteristic exponent of ξ such that $-\Psi(0) = q \geq 0$. In the case $\mathbb{Q}_x(T_0 < \infty) = 1$, Lamperti explained that for any $x > 0$, either $q > 0$ and the absorption time occurs by a jump, i.e.

$$\mathbb{Q}_x(X_{T_{0-}} > 0, T_0 < \infty) = 1,$$

or ξ drifts to $-\infty$ and T_0 occurs continuously, i.e.

$$\mathbb{Q}_x(X_{T_{0-}} = 0, T_0 < \infty) = 1.$$

We gather these two possibilities in the following:

$$\mathbf{H0}: \text{ Either } q > 0 \text{ or } E[\xi_1] < 0 \text{ and } q = 0.$$

Finally, from the Lamperti transformation (1), one gets the identity in distribution

$$T_0 \stackrel{(d)}{=} x^{\alpha} \Sigma_{\mathbf{e}_q}$$

where $\mathbf{e}_q$ is an independent exponential random variable of parameter q and where we understand that $\mathbf{e}_0 = \infty$.

Let $(U_t)_{t\geq 0}$ be the Ornstein-Uhlenbeck process associated to X defined by

$$U_t = e^{\tilde{\alpha} t} X_{(1-e^{-t})}, \; t \geq 0,$$

with $\tilde{\alpha} = \alpha^{-1}$. We write

$$H_0 = \inf\{s > 0; \; U_s = 0\}.$$

Theorem 1

For any $x > 0$ and $t > 0$, we have

$$\mathbb{Q}_x(T_0 < t) \quad = \quad \mathbb{Q}_{xt^{-\tilde{\alpha}}}(H_0 < \infty).$$

Let ξ be a spectrally negative Lévy process. Writing, when $q = 0$, $\psi(u) = \Psi(-iu), u \geq 0$, we have

$$\psi(u) = bu + \frac{\sigma}{2}u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur\mathbb{I}_{\{|r|<1\}})\nu(dr),$$

where $b \in \mathbb{R}$, $\sigma \geq 0$ and the measure ν satisfies the integrability condition $\int_{-\infty}^{0} (1 \wedge r^2)\,\nu(dr) < +\infty$. We write $\overline{\psi}(u) = \psi(u) - q$. It is well-known that

$$\lim_{u\to\infty} \frac{\psi(u)}{u} = \begin{cases} b > 0 & \text{if } \sigma = 0 \text{ and } \int_{-\infty}^{0} 1 \wedge |r|\nu(dr) < \infty \\ +\infty & \text{otherwise.} \end{cases} \quad (2)$$

If $\mathbb{E}[\xi_1] < 0$, then the equation $\psi(u) = 0$ admits 0 and $\theta > 0$ as roots. ψ has a well-defined inverse function $\phi : [0, \infty) \to [\max(\theta, 0), \infty)$.

Let ψ such that $\psi'(0^+) \geq 0$, set $a_0(\psi;\alpha) = 1$ and introduce the following entire function

$$\mathcal{I}_\psi(z) = \sum_{n=0}^{\infty} a_n(\psi;\alpha) z^n$$

where

$$a_n(\psi;\alpha) = \left(\prod_{k=1}^{n} \psi(\alpha k)\right)^{-1}$$

P.[08] suggested the following generalization, for any $\mathfrak{Re}(\kappa) > 0$,

$$\begin{aligned} \mathcal{I}_\psi(\kappa; z) &= \frac{1}{\Gamma(\kappa)} \int_0^\infty e^{-t} t^{\kappa-1} \mathcal{I}_\psi(tz)\, dt \\ &= \frac{1}{\Gamma(\kappa)} \sum_{n=0}^{\infty} a_n(\psi;\alpha) \Gamma(\kappa+n) z^n, \end{aligned}$$

where Γ stands for the Gamma function and the second identity is valid for any $|z| < \lim_{u\to\infty} \frac{\psi(\alpha u)}{u} > 0$.

Some examples

- Let ξ be a 2-scaled Brownian motion with drift $2b \in \mathbb{R}$, i.e $\psi(u) = 2u^2 + 2bu$. We obtain,

$$\mathcal{I}_\psi(\kappa; z) \quad = \quad \Phi\left(\kappa, b+1; z/2\right)$$

  where Φ stands for the confluent hypergeometric function.
- Let X be a regular spectrally negative α-stable process killed upon entering into the negative half-line. Caballero and Chaumont [06], (resp. Patie [09]) characterized the characteristic triplet (resp. the Laplace exponent) of the underlying Lévy process. Then,

$$\begin{aligned}\mathcal{I}_\psi(\kappa; z) &= \frac{1}{\Gamma(\kappa)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\kappa+1-\tilde{\alpha})}{\Gamma(\alpha n+\alpha)} z^n \\ &= {}_2\Psi_1\left(\begin{array}{c} (1,1),(1,\kappa+1-\tilde{\alpha}) \\ (\alpha,\alpha) \end{array} \middle| \, z \right)\end{aligned}$$

  where ${}_2\Psi_1$ stands for the Wright hypergeometric function.

- We consider the so-called saw-tooth process introduced and deeply studied by Carmona et al.[98]. It is a self-similar positive Markov process of index $\alpha = 1$ with underlying Lévy process the sum of a drift of parameter $b = 1$ and the negative of a compound Poisson process of parameter $\beta + \delta - 1 > 0$ whose jumps are exponentially distributed with parameter $\beta > 0$, i.e. $\psi_{1-\delta}(u) = u\frac{u+1-\delta}{u+\beta}$. Thus, for $|z| < 1$

$$\begin{aligned}\mathcal{I}_{\psi}(\kappa; z) &= \frac{\Gamma(2-\delta)}{\Gamma(\kappa)\Gamma(1+\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\kappa+n)\Gamma(n+1+\beta)}{\Gamma(n+2-\delta)n!} z^n \\ &= {}_2F_1\left(\kappa, 1+\beta, 2-\delta; z\right)\end{aligned}$$

where ${}_2F_1$ stands for the hypergeometric function.

Proposition

If $\lim_{u\to\infty} \frac{\psi(u)}{u} = b > 0$, then $\mathcal{I}_\psi(\kappa; z)$ is analytic in $|z| < \alpha b$ and for any fixed $\kappa = 0, -1, \ldots$, $\mathcal{I}_\psi(\kappa; z)$ is an entire function.
For any $\kappa \neq 0, -1, \ldots$ and any $\mathfrak{Re}(z) < \frac{\alpha b}{2}$, we have

$$\mathcal{I}_\psi(\kappa; z) = \left(1 - \frac{z}{\alpha b}\right)^{-\kappa} \sum_{n=0}^{\infty} \mathcal{I}_\psi(-n; \alpha b) \frac{\Gamma(\kappa + n)}{\Gamma(\kappa) n!} \left(\frac{z}{z - \alpha b}\right)^n.$$

In order to present our next result in a compact form, when **H0** holds, we write

$$\gamma = \begin{cases} \phi(q) & \text{if } q > 0, \\ \theta & \text{otherwise,} \end{cases} \tag{3}$$

and, for any $u \geq 0$,

$$\psi_\gamma(u) = \begin{cases} \overline{\psi}_{\phi(q)}(u) = \psi(u + \phi(q)) - q & \text{if } q > 0, \\ \psi_\theta(u) = \psi(u + \theta) & \text{otherwise.} \end{cases} \tag{4}$$

We observe that $\psi_\gamma(0) = 0$ and $\psi'_\gamma(0^+) > 0$.

Theorem 2

Assume that **H0** holds and write $S(tx^{-\alpha}) = \mathbb{Q}_x(T_0 \geq t),\ x, t > 0$. Then, there exists a constant $C_\gamma > 0$ such that

$$\mathcal{I}_{\psi_\gamma}(\tilde{\alpha}\gamma; -x) \sim \frac{x^{-\tilde{\alpha}\gamma}}{C_\gamma} \quad \text{as } x \to +\infty \tag{5}$$

and

$$S(t) = C_\gamma t^{-\tilde{\alpha}\gamma} \mathcal{I}_{\psi_\gamma}(\tilde{\alpha}\gamma; -t^{-1}),\ t > 0. \tag{6}$$

The law of T_0 is absolutely continuous with a density given by

$$s(t) = \tilde{\alpha}\gamma C_\gamma t^{-\tilde{\alpha}\gamma - 1} \mathcal{I}_{\psi_\gamma}(1 + \tilde{\alpha}\gamma; -t^{-1}),\ t > 0.$$

Corollary

We have

$$S(t) \sim C_\gamma t^{-\tilde{\alpha}\gamma} \quad \text{as } t \to \infty \tag{7}$$

and, for any $m = 0, 1 \ldots$,

$$s^{(m)}(t) \sim (-1)^m C_\gamma \frac{\Gamma(m+1+\tilde{\alpha}\gamma)}{\Gamma(\tilde{\alpha}\gamma)} t^{-\tilde{\alpha}\gamma-1-m} \quad \text{as } t \to \infty. \tag{8}$$

Corollary

Let us assume that $\lim_{u\to\infty} \frac{\psi(u)}{u} = \infty$. Then, for $m = 0$ or 1, the entire function $z \mapsto \mathcal{I}_{\psi_\gamma}(m + \tilde{\alpha}\gamma; z)$ has no real zeros.

Kesten's constants

Let us first recall that, in Patie [08], the Laplace transform of T_0, in the case $\mathbb{E}[\xi_1] < 0$, $q = 0$ and $\theta < \alpha$, is given by

$$\mathbb{E}_x\left[e^{-rT_0}\right] = \mathcal{N}_{\psi,\theta}(rx^{\alpha}),\ r, x > 0, \tag{9}$$

where

$$\mathcal{N}_{\psi,\theta}(r) = \mathcal{I}_{\psi}(r) - C(\theta)r^{\tilde{\alpha}\theta}\mathcal{I}_{\psi_\theta}(r)$$

and the positive constant $C(\theta)$ is characterized by

$$\mathcal{I}_{\psi}(r) \sim C(\theta)r^{\tilde{\alpha}\theta}\mathcal{I}_{\psi_\theta}(r) \quad \text{as } r \to \infty.$$

Next, let us write $\psi_\gamma(\alpha u) = (\alpha u)^2\bar{\varphi}_\gamma(\alpha u)$. Thus,

$$\bar{\varphi}_\gamma(\alpha u) = \frac{\bar{b}}{\alpha u} + \frac{\sigma}{2} + \int_0^{\infty} e^{-\alpha u r}\int_{-\infty}^{-r}\int_{-\infty}^{-s} e^{\gamma v}\nu_\gamma(dv)dsdr.$$

where $\bar{b} = b + \sigma\gamma + \int_{-\infty}^{0}\left(e^{\gamma r} - \mathbb{I}_{\{|r|<1\}}\right) r\nu(dr)$. Hence, $a_s(\bar{\varphi}_\gamma;\alpha)$ is a meromorphic function in the domain $\{s \in \mathbb{C};\ \mathfrak{Re}(s) > -\gamma - 1\}$ with simple poles at the points $s_k = -k - 1$ for $k = 0, 1, \ldots$ and $s_k > -\gamma - 1$.

As above, one may define the function

$$\begin{aligned} a_s(\psi_\gamma;\alpha) &= \frac{1}{\alpha^2\Gamma^2(s+1)} a_s(\bar{\varphi}_\gamma;\alpha) \\ &= \frac{1}{\alpha^2\Gamma^2(s+1)} \prod_{k=1}^{\infty} \frac{\bar{\varphi}_\gamma(\alpha(k+s+1))}{\bar{\varphi}_\gamma(\alpha k)}. \end{aligned}$$

Corollary

If $\lim_{u\to\infty} \frac{\psi(u)}{u} = b$ then $C_\gamma = \alpha^{\tilde{\alpha}\gamma} a_{-\tilde{\alpha}\gamma}(\varphi_\gamma;\alpha)$. Otherwise, we have, writing $\psi(\alpha u) = \alpha u \varphi(\alpha u)$,

$$C_\gamma = \begin{cases} \psi'_\gamma(0^+) & \text{if } \tilde{\alpha}\gamma = 1 \\ \alpha^n \psi'_\gamma(0^+) \left(\prod_{k=1}^n \varphi(\alpha k)\right)^{-1} & \text{if } \tilde{\alpha}\gamma = n+1,\ n = 1,2\ldots \\ \frac{\alpha^{2\tilde{\alpha}\gamma}}{\Gamma(1-\tilde{\alpha}\gamma)} a_{-\tilde{\alpha}\gamma}(\bar{\varphi}_\gamma;\alpha) & \text{otherwise.} \end{cases}$$

Finally, if $q = 0$ and $0 < \theta < \alpha$. Then, the constant C_θ is such that

$$\mathcal{I}_\psi(r) \sim \Gamma(1-\tilde{\alpha}\theta) C_\theta r^{\tilde{\alpha}\theta} \mathcal{I}_{\psi_\theta}(r) \quad \text{as } r \to \infty.$$