The upper envelope of positive self-similar Markov processes.

Juan Carlos Pardo

University of Bath

Angers, 20 july 2009

Outline of the talk

Positive self-similar Markov processes (PSSMP).

The upper envelope of PSSMP.

The regular case.

The log-regular case.

Positive self-similar Markov processes

Positive self-similar Markov processes (PSSMP).

Definition

A $\mathbb{R}_+$-valued Markov process $X = (X_t, t \geq 0)$ with càdlàg paths is a self-similar process if for every $k > 0$ and every initial state $x \geq 0$ it satisfies the scaling property, i.e., for some $\alpha > 0$

$$\textit{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{kx},$$

where $\mathbb{P}_x$ denotes the law of the process X starting from $x \geq 0$.

Positive self-similar Markov processes (PSSMP).

Definition

A $\mathbb{R}_+$-valued Markov process $X = (X_t, t \geq 0)$ with càdlàg paths is a self-similar process if for every $k > 0$ and every initial state $x \geq 0$ it satisfies the scaling property, i.e., for some $\alpha > 0$

$$\textit{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{kx},$$

where $\mathbb{P}_x$ denotes the law of the process X starting from $x \geq 0$.

We denote by $X^{(x)}$ a PSSMP starting at $x \geq 0$.

Positive self-similar Markov processes (PSSMP).

Definition

A $\mathbb{R}_+$-valued Markov process $X = (X_t, t \geq 0)$ with càdlàg paths is a self-similar process if for every $k > 0$ and every initial state $x \geq 0$ it satisfies the scaling property, i.e., for some $\alpha > 0$

$$\text{the law of } (kX_{k^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{kx},$$

where $\mathbb{P}_x$ denotes the law of the process X starting from $x \geq 0$.

We denote by $X^{(x)}$ a PSSMP starting at $x \geq 0$.

Examples : Bessel processes, stable subordinators or more generally, stable processes conditioned to stay positive.

Lamperti representation.

Let $X^{(x)}$ be a self-similar Markov process started from $x > 0$ that fulfills the scaling property for some $\alpha > 0$, then

$$X_t^{(x)} = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\}, \qquad 0 \le t \le x^{\alpha} I(\xi),$$

where,

$$\tau_t = \inf\left\{s \ge 0 : I_s(\xi) > t\right\}, \quad I_s(\xi) = \int_0^s \exp\left\{\alpha \xi_u\right\} \mathrm{d}u,$$

$$I(\xi) = \lim_{t \to +\infty} I_t(\xi),$$

and ξ is a real Lévy process possibly killed at an independent exponential time.

The limit process $X^{(0)}$.

Let P be a reference probability measure on $\mathcal{D}$ under which ξ is a Lévy process and $H = (H_t, t \geq 0)$ its corresponding ascending ladder height process.

The limit process $X^{(0)}$.

Let P be a reference probability measure on $\mathcal{D}$ under which ξ is a Lévy process and $H = (H_t, t \geq 0)$ its corresponding ascending ladder height process.

Hypothesis

The process H is non arithmetic and $E(H_1) < \infty$

The limit process $X^{(0)}$.

Let P be a reference probability measure on $\mathcal{D}$ under which ξ is a Lévy process and $H = (H_t, t \geq 0)$ its corresponding ascending ladder height process.

Hypothesis

The process H is non arithmetic and $E(H_1) < \infty$

Chaumont et al. (2008) proved that under the above condition the family $\{X^{(x)}, x > 0\}$ converges weakly on $\mathcal{D}$ towards a PSSMP starting from 0 and with the same transition probabilities as $X^{(x)}$, $x > 0$. We denote this limit process by $X^{(0)}$ and its law by $\mathbb{P}_0$.

First and last passage times.

For $x \geq 0$, we define the first and last passage times of the process $X^{(0)}$ as follows :

$$S_x = \inf\{t \geq 0 : X^{(0)} \geq x\} \quad \text{and} \quad U_x = \sup\{t \geq 0 : X^{(0)} \leq x\}.$$

First and last passage times.

For $x \geq 0$, we define the first and last passage times of the process $X^{(0)}$ as follows :

$$S_x = \inf\{t \geq 0 : X^{(0)} \geq x\} \quad \text{and} \quad U_x = \sup\{t \geq 0 : X^{(0)} \leq x\}.$$

The processes $S = (S_x, x \geq 0)$ and $U = (U_x, x \geq 0)$ are increasing self-similar processes with scaling index $1/\alpha$.

First and last passage times.

For $x \geq 0$, we define the first and last passage times of the process $X^{(0)}$ as follows :

$$S_x = \inf\{t \geq 0 : X^{(0)} \geq x\} \quad \text{and} \quad U_x = \sup\{t \geq 0 : X^{(0)} \leq x\}.$$

The processes $S = (S_x, x \geq 0)$ and $U = (U_x, x \geq 0)$ are increasing self-similar processes with scaling index $1/\alpha$.

If $X^{(0)}$ has no positive jumps, then S and U have independent increments.

PSSMP with no positive jumps.

Proposition

i) *Let* $E(\xi_1) := m \geq 0$. *For every* $x > 0$, *the law of* S_x *is the same as that of*

$$x^{\alpha}\int_0^{\infty}\exp\{-\alpha\xi_u^{\uparrow}\}\mathrm{d}u,$$

where $\xi^{\uparrow}$ *is the Lévy process conditioned to stay positive.*

PSSMP with no positive jumps.

Proposition

i) *Let $E(\xi_1) := m \geq 0$. For every $x > 0$, the law of S_x is the same as that of*

$$x^{\alpha} \int_0^{\infty} \exp\{-\alpha \xi_u^{\uparrow}\} \mathrm{d}u,$$

where $\xi^{\uparrow}$ is the Lévy process conditioned to stay positive.

ii) *Let $m > 0$. For every $x > 0$, the law of U_x is the same as that of*

$$x^{\alpha} \int_0^{\infty} \exp\{-\alpha \xi_u\} \mathrm{d}u.$$

The upper envelope of positive self-similar Markov processes.

The upper envelope.

Let $\mathcal{H}_0$ be the set of all positive increasing functions $h(t)$ on $(0, +\infty)$ satisfying

i) $h(0) = 0$,

ii) there exist $\beta \in (0, 1)$ such that $\sup_{t<\beta} \frac{t}{h(t)} < \infty$.

The upper envelope.

Let $\mathcal{H}_0$ be the set of all positive increasing functions $h(t)$ on $(0,+\infty)$ satisfying

i) $h(0) = 0$,

ii) there exist $\beta \in (0,1)$ such that $\sup_{t<\beta} \frac{t}{h(t)} < \infty$.

For the study at $+\infty$, we define $\mathcal{H}_\infty$, the set of all positive increasing functions $h(t)$ on $(0,+\infty)$ satisfying

i) $\lim_{t\to\infty} h(t) = \infty$,

ii) there exists, $\beta > 1$ such that $\sup_{t>\beta} \frac{t}{h(t)} < \infty$.

The upper envelope

We also define,

$$G(t) = \mathbb{P}(S_1 < t), \qquad F(t) = \mathbb{P}(I(-\xi) < t),$$

and

$$F^{\uparrow}(t) = \mathbb{P}(I(-\xi^{\uparrow}) < t).$$

The upper envelope

We also define,

$$G(t) = \mathbb{P}(S_1 < t), \qquad F(t) = \mathbb{P}(I(-\xi) < t),$$

and

$$F^{\uparrow}(t) = \mathbb{P}(I(-\xi^{\uparrow}) < t).$$

Remark

With no loss of generality, we will suppose that $\alpha = 1$. Indeed, we see from the scaling property that if $X^{(x)}$, $x \geq 0$, is a PSSMP with index $\alpha > 0$, then $\left(X^{(x)}\right)^{\alpha}$ is a PSSMP with index equal to 1. Therefore, the integral tests and LIL established in the sequel can easily be interpreted for any $\alpha > 0$.

The case with no positive jumps.

In this case, we assume that

$$\mathbb{E}\left(\log^+ I(-\xi^{\uparrow})^{-1}\right) < \infty.$$

The case with no positive jumps.

In this case, we assume that

$$\mathbb{E}\left(\log^{+} I(-\xi^{\uparrow})^{-1}\right) < \infty.$$

Theorem (1)

Let $m \geq 0$ *and* $h \in \mathcal{H}_0$,

The case with no positive jumps.

In this case, we assume that

$$\mathbb{E}\left(\log^+ I(-\xi^{\uparrow})^{-1}\right) < \infty.$$

Theorem (1)

Let $m \geq 0$ *and* $h \in \mathcal{H}_0$*,*

i) *if*

$$\int_{0+} F^{\uparrow}\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$*,*

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t), \; i.o., \; as \; t \to 0\Big) = 0.$$

The case with no positive jumps.

Theorem (1)

The case with no positive jumps.

Theorem (1)

ii) *if*

$$\int_{0+} F^{\uparrow}\left(\frac{t}{h(t)}\right)\frac{dt}{t} = \infty,$$

then for every $\epsilon > 0$,

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t),\ i.o.,\ as\ t \to 0\Big) = 1.$$

The case with no positive jumps.

Theorem (2)

Let $m \geq 0$ *and* $h \in \mathcal{H}_\infty$,

The case with no positive jumps.

Theorem (2)

Let $m \geq 0$ *and* $h \in \mathcal{H}_\infty$*,*

i) *if*

$$\int^{+\infty} F^{\uparrow}\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$*,*

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 0.$$

The case with no positive jumps.

Theorem (2)

Let $m \geq 0$ *and* $h \in \mathcal{H}_\infty$,

i) *if*

$$\int^{+\infty} F^{\uparrow}\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$,

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t),\ \ i.o.,\ \ as\ t \to +\infty\Big) = 0.$$

ii) *if*

$$\int^{+\infty} F^{\uparrow}\left(\frac{t}{h(t)}\right)\frac{dt}{t} = \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$,

$$\mathbb{P}_x\Big(X_t > (1-\epsilon)h(t),\ \ i.o.,\ \ as\ t \to +\infty\Big) = 1.$$

The general case.

Theorem (3)

Let $m > 0$ *and* $h \in \mathcal{H}_0$,

The general case.

Theorem (3)

Let $m > 0$ *and* $h \in \mathcal{H}_0$,

i) *if*

$$\int_0 G\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$,

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t),\ i.o.,\ as\ t \to 0\Big) = 0.$$

The general case.

Theorem (3)

Let $m > 0$ *and* $h \in \mathcal{H}_0$,

i) *if*

$$\int_0 G\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$,

$$\mathbb{P}_0\Big(X_t > (1+\epsilon)h(t),\ \ i.o.,\ as\ t \to 0\Big) = 0.$$

ii) *if*

$$\int_0 F\left(\frac{t}{h(t)}\right)\frac{dt}{t} = \infty,$$

then for every $\epsilon > 0$,

$$\mathbb{P}_0\Big(X_t > (1-\epsilon)h(t),\ \ i.o.,\ as\ t \to 0\Big) = 1.$$

The general case.

Theorem (4)

Let $m > 0$ *and* $h \in \mathcal{H}_\infty$,

The general case.

Theorem (4)

Let $m > 0$ *and* $h \in \mathcal{H}_\infty$,

i) *if*

$$\int^{+\infty} G\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$,

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 0.$$

The general case.

Theorem (4)

Let $m > 0$ *and* $h \in \mathcal{H}_\infty$*,*

i) *if*

$$\int^{+\infty} G\left(\frac{t}{h(t)}\right)\frac{dt}{t} < \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$*,*

$$\mathbb{P}_x\Big(X_t > (1+\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 0.$$

ii) *if*

$$\int^{+\infty} F\left(\frac{t}{h(t)}\right)\frac{dt}{t} = \infty,$$

then for every $\epsilon > 0$ *and every* $x \geq 0$*,*

$$\mathbb{P}_x\Big(X_t > (1-\epsilon)h(t),\ i.o.,\ as\ t \to +\infty\Big) = 1.$$

The regular case.

The regular case.

Let ν an independent r.v. of $I(-\xi)$ and such that $0 < \nu \leq 1$ and set

$$F_\nu(t) = \mathbb{P}(\nu I(-\xi) < t).$$

The regular case.

Let ν an independent r.v. of $I(-\xi)$ and such that $0 < \nu \leq 1$ and set

$$F_\nu(t) = \mathbb{P}(\nu I(-\xi) < t).$$

Suppose that $m > 0$ and that

$$ct^\beta L(t) \leq F(t) \leq F_\nu(t) \leq Ct^\beta L(t) \qquad \text{as} \quad t \to 0, \qquad (3.1)$$

where $\beta > 0$, c and C are two positive constants such that $c \leq C$, and L is a slowly varying function at 0.

The regular case.

Let ν an independent r.v. of $I(-\xi)$ and such that $0 < \nu \leq 1$ and set

$$F_\nu(t) = \mathbb{P}(\nu I(-\xi) < t).$$

Suppose that $m > 0$ and that

$$ct^\beta L(t) \leq F(t) \leq F_\nu(t) \leq Ct^\beta L(t) \qquad \text{as} \quad t \to 0, \tag{3.1}$$

where $\beta > 0$, c and C are two positive constants such that $c \leq C$, and L is a slowly varying function at 0.

Proposition

Under condition (3.1), *we have*

$$ct^\beta L(t) \leq G(t) := \mathbb{P}(S_1 < t) \leq C_\epsilon t^\beta L(t) \qquad as \quad t \to 0$$

where C_ϵ is a positive constant bigger than C.

The regular case.

Theorem (5)

Under condition (3.1), the upper envelope of the PSSMP at 0 and at $+\infty$ is as follows :

The regular case.

Theorem (5)

Under condition (3.1), the upper envelope of the PSSMP at 0 *and at* $+\infty$ *is as follows :*

i) *Let* $h \in \mathcal{H}_0$*, such that either* $\lim_{t\to 0} t/h(t) = 0$ *or* $\liminf_{t\to 0} t/h(t) > 0$*, then*

$$\mathbb{P}\Big(X_t^{(0)} > h(t),\ \ i.o.,\ as\ t \to 0\Big) = 0\ \ or\ 1,$$

accordingly as $\int_{0+} F\,(t/h(t))\, t^{-1}\mathrm{d}t$ *is finite or infinite.*

The regular case.

Theorem (5)

ii) *Let* $h \in \mathcal{H}_\infty$*, such that either* $\lim_{t\to+\infty} t/h(t) = 0$ *or* $\liminf_{t\to+\infty} t/h(t) > 0$*, then for all* $x \geq 0$

$$\mathbb{P}\Big(X_t^{(x)} > h(t),\ i.o.,\ as\ t \to \infty\Big) = 0\ or\ 1,$$

accordingly as $\int^{+\infty} F\left(t/h(t)\right) t^{-1}\mathrm{d}t$ *is finite or infinite.*

The log-regular case.

The log-regular case.

Suppose that

$$-\log F_\nu(1/t) \sim -\log F(1/t) \sim \lambda t^\beta L(t), \text{ as} t \to +\infty, \qquad (4.2)$$

where $\lambda > 0$, $\beta > 0$ and L is a slowly varying function at $+\infty$.

The log-regular case.

Suppose that

$$-\log F_{\nu}(1/t) \sim -\log F(1/t) \sim \lambda t^{\beta} L(t), \text{ as} t \to +\infty, \quad (4.2)$$

where $\lambda > 0$, $\beta > 0$ and L is a slowly varying function at $+\infty$.

Proposition

Under condition (4.2), *we have*

$$-\log G(1/t) \sim \lambda t^{\beta} L(t), \qquad as \quad t \to +\infty.$$

Examples.

Examples.

- Bessel processes of dimension $\delta \geq 2$.

Examples.

- Bessel processes of dimension $\delta \geq 2$.
- Stable Lévy processes with no positive jumps conditioned to stay positive.

Examples.

- Bessel processes of dimension $\delta \geq 2$.
- Stable Lévy processes with no positive jumps conditioned to stay positive.
- PSSMP associated by the Lamperti representation to Lévy processes that drift to $+\infty$ and that have exponential moments of arbitrary positive order.

Examples.

- Bessel processes of dimension $\delta \geq 2$.
- Stable Lévy processes with no positive jumps conditioned to stay positive.
- PSSMP associated by the Lamperti representation to Lévy processes that drift to $+\infty$ and that have exponential moments of arbitrary positive order.
- PSSMP with no positive jumps satisfying that the Laplace exponents of the first and last passage times are regularly varying.

The log-regular case

Define the function

$$\phi(t) := t \inf \left\{ s : 1/F(1/s) > |\log t| \right\}, \quad t > 0, \quad t \neq 1.$$

The log-regular case

Define the function

$$\phi(t) := t \inf \left\{ s : 1/F(1/s) > |\log t| \right\}, \quad t > 0, \quad t \neq 1.$$

Theorem (6)

Under condition (4.2), we have the following law of the iterated logarithm :

The log-regular case

Define the function

$$\phi(t) := t \inf\left\{s : 1/F(1/s) > |\log t|\right\}, \quad t > 0, \quad t \neq 1.$$

Theorem (6)

Under condition (4.2), we have the following law of the iterated logarithm :

i)

$$\limsup_{t \to 0} \frac{X_t^{(0)}}{\phi(t)} = 1, \qquad \textit{almost surely.}$$

ii) *For all* $x \geq 0$,

$$\limsup_{t \to +\infty} \frac{X_t^{(x)}}{\phi(t)} = 1, \qquad \textit{almost surely.}$$

The log-regular case

Now, we suppose that

$$-\log F_\nu(1/t) \sim -\log F(1/t) \sim K(\log t)^\gamma, \text{ as } t \to +\infty, \quad (4.3)$$

where $K > 0$ and $\gamma > 0$.

The log-regular case

Now, we suppose that

$$-\log F_\nu(1/t) \sim -\log F(1/t) \sim K(\log t)^\gamma, \text{ as } t \to +\infty, \quad (4.3)$$

where $K > 0$ and $\gamma > 0$.

Proposition

Under condition (4.3), *we have*

$$-\log G(1/t) \sim K(\log t)^\gamma, \qquad \textit{as} \quad t \to 0.$$

The log-regular case

Now, we suppose that

$$-\log F_\nu(1/t) \sim -\log F(1/t) \sim K(\log t)^\gamma, \text{ as } t \to +\infty, \quad (4.3)$$

where $K > 0$ and $\gamma > 0$.

Proposition

Under condition (4.3), *we have*

$$-\log G(1/t) \sim K(\log t)^\gamma, \qquad \text{as} \quad t \to 0.$$

Example : The PSSMP associated by the Lamperti representation to the Poisson process.

Le cas log-régulier.

Define the function

$$\Phi(t) := t \exp\left\{(K^{-1} \log|\log t|)^{1/\gamma}\right\}, \quad t > 0, \quad t \neq 1.$$

Le cas log-régulier.

Define the function

$$\Phi(t) := t \exp\left\{(K^{-1}\log|\log t|)^{1/\gamma}\right\}, \quad t > 0, \quad t \neq 1.$$

Theorem (7)

Under condition (4.3), we have the following law of the iterated logarithm :

Le cas log-régulier.

Define the function

$$\Phi(t) := t \exp\left\{(K^{-1}\log|\log t|)^{1/\gamma}\right\}, \quad t > 0, \quad t \neq 1.$$

Theorem (7)

Under condition (4.3), we have the following law of the iterated logarithm :

i)

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\Phi(t)} = 1, \qquad \textit{almost surely.}$$

ii) *Pour tout* $x \geq 0$,

$$\limsup_{t\to +\infty} \frac{X_t^{(x)}}{\Phi(t)} = 1, \qquad \textit{almost surely.}$$

The case with no positive jumps.

Theorem (8)

Suppose that for all $x \geq 0$

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\Lambda(t)} = 1 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{X_t^{(x)}}{\Lambda(t)} = 1 \quad a.s.,$$

where Λ *is a positive function such that* $\Lambda(0) = 0$ *and* $\lim_{t\to\infty} \Lambda(t) = \infty$*, then*

The case with no positive jumps.

Theorem (8)

Suppose that for all $x \geq 0$

$$\limsup_{t\to 0} \frac{X_t^{(0)}}{\Lambda(t)} = 1 \quad \text{and} \quad \limsup_{t\to\infty} \frac{X_t^{(x)}}{\Lambda(t)} = 1 \quad a.s.,$$

where Λ *is a positive function such that* $\Lambda(0) = 0$ *and* $\lim_{t\to\infty} \Lambda(t) = \infty$, *then*

i) *for all* $x \geq 0$

$$\limsup_{t\to 0} \frac{J_t^{(0)}}{\Lambda(t)} = 1 \quad \text{and} \quad \limsup_{t\to\infty} \frac{J_t^{(x)}}{\Lambda(t)} = 1 \quad a.s.,$$

where $J_t^{(x)} = \inf_{s\geq t} X_s^{(x)}$.

The case with no positive jumps.

Theorem (8)

ii) *for all* $x \geq 0$

$$\limsup_{t\to 0} \frac{X_t^{(0)} - J_t^{(0)}}{\Lambda(t)} = 1 \quad and \quad \limsup_{t\to\infty} \frac{X_t^{(x)} - J_t^{(x)}}{\Lambda(t)} = 1 \quad a.s.$$