Combining Stein's method with Malliavin calculus, part I

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ISONORMAL GAUSSIAN PROCESS

We consider a real-valued centered isonormal Gaussian process

$$X = \{X(h) : h \in \mathfrak{H}\},$$

where $\mathfrak{H}$ is a separable Hilbert space, and we assume that

$$E\Big(X(h)\,X(g)\Big) = \langle h, g\rangle_{\mathfrak{H}} \quad (\langle \cdot, \cdot \rangle_{\mathfrak{H}} = \text{inner product on } \mathfrak{H}).$$

For instance: $\mathfrak{H} = L^2([0,T])$ corresponds to the case where X is a standard Brownian motion on $[0,T]$ (using the convention $X_t = X(\mathbf{1}_{[0,t]})$).

WIENER CHAOS

For $q \geq 0$, $\mathcal{H}_q$ denotes the qth Wiener chaos associated with X.

That is, $\mathcal{H}_q$ is the L^2-closed vector space generated by r.v. of the type $H_q(X(h))$, where H_q is the Hermite polynomial of degree q and $h \in \mathfrak{H}$ is such that $\mathrm{Var}(X(h)) = \|h\|_{\mathfrak{H}}^2 = 1$.

We have $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, and so on.

Example: If B^H is a fractional Brownian motion of Hurst index $H \in (0,1)$ then

$$Z_T = \int_0^T H_q(B_{u+1}^H - B_u^H)du$$

is an element of the qth chaos of B^H.

AN ILLUSTRATING EXAMPLE

Let

$$Z_T = \int_0^T H_q(B^H_{u+1} - B^H_u)du.$$

When $H \leq 1 - 1/(2q)$ (necessary and sufficient condition, recall Taqqu's talk), how to prove that

$$\frac{Z_T}{\sqrt{\mathrm{Var}(Z_T)}}$$

converges in law to $N(0,1)$ as $T \to \infty$?

Historical method: Breuer-Major, Dobrushin-Major, Giraitis-Surgailis, Taqqu, . . . (in the eighties) used the so-called "method of moments".

OUR RESULT

Theorem (Nourdin, Peccati, Reinert) Let Z be an element of the qth Wiener chaos of some isonormal Gaussian process X. Assume that $\mathrm{Var}(Z) = E(Z^2) = 1$. Then, for $N \sim N(0,1)$,

$$\boxed{\sup_{z\in\mathbb{R}} \left| P(Z \le z) - P(N \le z) \right| \le \sqrt{\tfrac{q-1}{3q}\left| E(Z^4) - 3 \right|}}.$$

A FIRST CONSEQUENCE

As a first consequence, we recover (and refine) a result which is at the very beginning of all this story, that is the so-called *Nualart and Peccati criterion of asymptotic normality*:

Theorem (Nualart, Peccati) Fix $q \geq 2$, and let (Z_T) be a family belonging to the qth Wiener chaos of some isonormal Gaussian process X. Assume that $\mathrm{Var}(Z_T) \to 1$. Then, as $T \to \infty$,

$$\boxed{Z_T \to N(0,1) \text{ in law if and only if } E(Z_T^4) \to 3.}$$

Observe the drastic simplification wrt the method of moments!

FIRST INGREDIENT FOR THE PROOF

Stein's method

STEIN'S METHOD : Let $N \sim N(0,1)$, and fix $z \in \mathbb{R}$. Define the function $f_z : \mathbb{R} \to \mathbb{R}$ by

$$f_z(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} \Big(\mathbf{1}_{(-\infty,z]}(a) - P(N \leq z) \Big) e^{-\frac{a^2}{2}} da.$$

Then f_z is bounded on $\mathbb{R}$, continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{z\}$. Moreover, it verifies

$$\mathbf{1}_{(-\infty,z]}(x) - P(N \leq z) = f_z'(x) - x f_z(x), \quad x \in \mathbb{R} \setminus \{z\},$$

and it is Lipschitz with constant bounded by 1.

We deduce that, for <u>any</u> random variable Z:

$$\boxed{\sup_{z\in\mathbb{R}} \Big| P(Z \leq z) - P(N \leq z) \Big| \leq \sup_{\|\varphi\|_{\text{Lip}} \leq 1} \Big| E[\varphi'(Z)] - E[Z\varphi(Z)] \Big|.}$$

QUESTION: How to estimate

$$\sup_{\|\varphi\|_{\mathsf{Lip}}\leq 1} \Big| E[\varphi'(Z)] - E[Z\varphi(Z)] \Big| ?$$

Of course, it will depend on the *specific form* of Z. Several cases are well studied in the literature: exchangeable pairs, diffusion generators, size/zero-bias transforms, local dependency graphs, etc.

HERE: We investigate what happens when, for Z, we choose an element of the qth Wiener chaos ($q \geq 2$) of a given isonormal Gaussian process X.

SECOND INGREDIENT FOR THE PROOF

Malliavin calculus

From now on, **ONLY** for the simplicity of exposition, we assume that $X = B$ is the standard Brownian motion on $[0, T]$.

Important fact: Any r.v. Z belonging to the qth Wiener chaos of B is equal to a q-times iterated Itô's integral as follows:

$$Z = q! \int_0^T dB_{t_1} \ldots \int_0^{t_{q-2}} dB_{t_{q-1}} \int_0^{t_{q-1}} dB_{t_q} \, f(t_1, \ldots, t_q) =: I_q(f)$$

Here, the function $f : [0, T]^q \to \mathbb{R}$ is *symmetric* and *square integrable* (and is also unique).

MALLIAVIN DERIVATIVE. When $Z = I_q(f)$, its Malliavin derivative is given by

$$D_t Z = D_t I_q(f) = q\, I_{q-1}\Big(f(\cdot,t)\Big), \quad t \in [0,T].$$

Integration by parts formula. We have, for $Z = I_q(f)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz:

$$\boxed{E[Z\varphi(Z)] = E[\varphi'(Z) \times \tfrac{1}{q}\|DZ\|^2].}$$

Here, $\|\cdot\|$ is a shorthand notation for $\|\cdot\|_{L^2([0,T])}$

COMBINING STEIN'S METHOD AND MALLIAVIN CALCULUS

Assume that $Z = I_q(f)$ (with $f : [0,T]^q \to \mathbb{R}$ symmetric and square integrable), and let $N \sim N(0,1)$. We have

$$
\begin{aligned}
\sup_{z\in\mathbb{R}} \Big| P(Z \le z) - P(N \le z) \Big| &\le \sup_{\|\varphi\|_{\mathrm{Lip}}\le 1} \Big| E[Z\varphi(Z)] - E[\varphi'(Z)] \Big| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\le 1} \Big| E\Big[\varphi'(Z)\Big(\frac{1}{q}\|DZ\|^2 - 1\Big)\Big] \Big| \\
&\le E\left[\Big|\frac{1}{q}\|DZ\|^2 - 1\Big|\right] \\
&\le \sqrt{E\left[\Big(\frac{1}{q}\|DZ\|^2 - 1\Big)^2\right]}.
\end{aligned}
$$

PROOF OF THEOREM

To get what we want, that is

$$\sup_{z\in\mathbb{R}}\Big|P(Z\le z)-P(N\le z)\Big|\le\sqrt{\frac{q-1}{3q}\Big|E(Z^4)-3\Big|},$$

we are left to prove that

$$E\left[\Big(\frac{1}{q}\|DZ\|^2-1\Big)^2\right]\le\frac{q-1}{3q}\,\Big|E[Z^4]-3\Big|.$$

Key tool: the product formula between multiple integrals

Product formula. If $f : [0,1]^p \to \mathbb{R}$ and $g : [0,1]^q \to \mathbb{R}$ are symmetric and square-integrable, then

$$\boxed{I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g),}$$

Here $f\widetilde{\otimes}_r g$ denotes the symmetrization of $f \otimes_r g$ defined by

$$\begin{aligned} f \otimes_r g(t_1, \ldots, t_{p+q-2r}) &= \int_{[0,T]^r} f(t_1, \ldots, t_{p-r}, x_1, \ldots, x_r) \\ &\quad \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, x_1, \ldots, x_r) dx_1 \ldots dx_r. \end{aligned}$$

In words, you identify r variables in f and g and you integrate them out.

Isometry and orthogonality formulas. If $f : [0,1]^p \to \mathbb{R}$ and $g : [0,1]^q \to \mathbb{R}$ are symmetric and square-integrable, then

$$\boxed{E[I_p(f)I_q(g)] = q!\langle f, g\rangle \text{ if } p = q, \text{ and } E[I_p(f)I_q(g)] = 0 \text{ otherwise.}}$$

Application to our problem, I. Let $Z = I_q(f)$ with $\mathrm{Var}(Z) = 1$. We have $D_t Z = q I_{q-1}\big(f(\cdot,t)\big)$. We can write

$$\begin{aligned}
\frac{1}{q}\|DZ\|^2 &= q\int_0^T I_{q-1}\big(f(\cdot,t)\big)^2 dt \\
&= q\int_0^T \sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}\big(f(\cdot,t)\widetilde{\otimes}_r f(\cdot,t)\big)dt \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}\Big(\int_0^T f(\cdot,t)\widetilde{\otimes}_r f(\cdot,t)dt\Big) \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I_{2q-2-2r}(f\widetilde{\otimes}_{r+1} f) \\
&= 1 + q\sum_{r=1}^{q-1} (r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}(f\widetilde{\otimes}_r f).
\end{aligned}$$

By orthogonality and isometry:

$$E\left[\Big(\frac{1}{q}\|DZ\|^2-1\Big)^2\right]=q^2\sum_{r=1}^{q-1}(r-1)!^2\binom{q-1}{r-1}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2.$$

Application to our problem, II. Let $Z = I_q(f)$ with $\mathrm{Var}(Z) = 1$. First, we have

$$E[Z^4] = E[Z \times Z^3] = 3E[Z^2 \times \frac{1}{q}\|DZ\|^2].$$

Second, we have

$$Z^2 = I_q(f)^2 = \sum_{r=0}^{q} r!\binom{q}{r}^2 I_{2q-2r}(f\widetilde{\otimes}_r f).$$

Since

$$\frac{1}{q}\|DZ\|^2 = 1 + q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I_{2q-2r}(f\widetilde{\otimes}_r f),$$

we deduce

$$E[Z^4] - 3 = \frac{3}{q}\sum_{r=1}^{q-1} rr!^2\binom{q}{r}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2.$$

THE END. Using

$$E\left[\left(\frac{1}{q}\|DZ\|^2-1\right)^2\right]=q^2\sum_{r=1}^{q-1}(r-1)!^2\binom{q-1}{r-1}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2$$

and

$$E[Z^4]-3=\frac{3}{q}\sum_{r=1}^{q-1}rr!^2\binom{q}{r}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2,$$

we easily prove that

$$E\left[\left(\frac{1}{q}\|DZ\|^2-1\right)^2\right]\leq\sqrt{\frac{q-1}{3q}\Big(E(Z^4)-3\Big)},$$

and the proof of the theorem is done.

APPLICATION TO FBM. Let $N \sim N(0,1)$. Let $H \leq 1 - \frac{1}{2q}$, and set $Z_T = \int_0^T H_q(B^H_{u+1} - B^H_u)du$. We have

$$\sup_{z\in\mathbb{R}} \left| P\left(\frac{Z_T}{\sqrt{\mathrm{Var}(Z_T)}} \leq z \right) - P(N \leq z) \right|$$

$$\leq \mathrm{cst} \times \begin{cases} T^{-\frac{1}{2}} & \text{if } H \leq \frac{1}{2} \\ T^{H-1} & \text{if } \frac{1}{2} \leq H \leq \frac{2q-3}{2q-2} \\ T^{qH-q+\frac{1}{2}} & \text{if } \frac{2q-3}{2q-2} \leq H < 1 - \frac{1}{2q} \\ (\log T)^{-\frac{1}{2}} & \text{if } H = 1 - \frac{1}{2q}. \end{cases}$$

ALSO AVAILABLE (in a similar fashion):

(i) *Multidimensional version*. See Peccati and Tudor (without bound) and Nourdin, Peccati and Réveillac (with bounds).

(ii) *Other limit laws than* $N(0,1)$.

χ^2 CASE

When $\nu \geq 1$ is an integer, we define the χ_c^2 law (of ν degrees of freedom) as the law of $\sum_{i=1}^{\nu}(N_i^2 - 1)$, where the N_i's are iid $N(0,1)$.

Theorem (Nourdin, Peccati). Fix $q \geq 2$. Let (Z_n) be a sequence belonging to the qth Wiener chaos of some isonormal Gaussian process X. Assume that $\mathrm{Var}(Z_n) \to 2\nu$. Then, as $n \to \infty$,

$$Z_n \to \chi_c^2(\nu) \text{ in law if and only if } E(Z_n^4) - 12E(Z_n^3) \to 12\nu^2 - 48\nu.$$

(We also have a version with bounds.)

SOME REFERENCES (can be downloaded in our respective webpages):

(a) I. Nourdin and G. Peccati: "Stein's method meets Malliavin calculus: a short survey with new estimates"

(b) I. Nourdin: "Integration by parts on Wiener space and applications" (notes of a doctoral course delivered in Nov. 2008 at CMM, Santiago de Chile)

(c) G. Peccati: "Stein's method, Malliavin calculus and infinite-dimensional Gaussian analysis" (notes of a series of lectures delivered in Jan. 2009 at Singapore)