Scaling limits of random maps with large faces

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Selfsimilar Processes and their Applications
Angers, July 20–24, 2009

Planar maps

Definition

A planar map is a proper embedding of a connected graph in the two-dimensional sphere, considered up to direct homeomorphisms of the sphere.

- A rooted map: an oriented edge (e) is distinguished
- A pointed map: a vertex (v_*) is distinguished
- Notations:
  - $V(\mathbf{m})$ set of vertices
  - $F(\mathbf{m})$ set of faces
  - d_{gr} the graph distance

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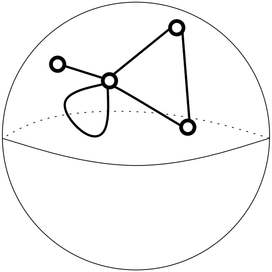

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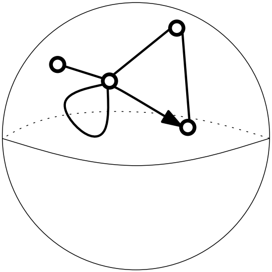

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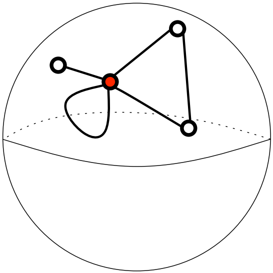

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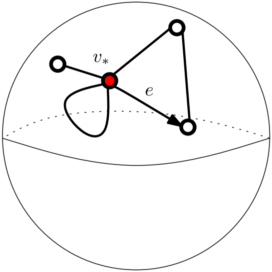

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General motivation

Maps appear naturally in many contexts

- Graph theory (4-color theorem, embedability)
- Counting problems
  - by direct resolution of the equation soled by generating functions [Tutte]
  - by using matrix integrals [t'Hooft, Brézin-Parisi-Itzykson-Zuber]
  - by algebraic methods: representation theory of the symmetric group, algebraic geometry [Harer-Zagier, Goulden-Jackson]
  - by bijective methods [Cori-Vauquelin, Schaeffer, Bouttier-Di Francesco-Guitter]
- Theoretical physics: random maps are natural models of random surfaces (discretization of 2D quantum gravity)
- Probability theory: finding scaling limits for discrete 'combinatorial' random structures (e.g. Donsker's theorem, continuum random trees, statistical physics in 2D and SLE)

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Natural ways of picking a map at random

All maps we consider are rooted.

- pick a p-angulation with n vertices, uniformly at random (ex $p = 3$ triangulation, $p = 4$ quadrangulation)
- From now on we only consider bipartite plane maps (with faces of even degree)
- Boltzmann distribution: let $q = (q_k, k \geq 1)$ a non-negative sequence. Define a measure on the set of (rooted) planar maps by

$$W_q(\mathbf{m}) = \prod_{f \in F(\mathbf{m})} q_{\deg(f)/2} \cdot$$

- Let

$$W_q^n(\cdot) = W_q\Big(\cdot \,\Big|\, \{\mathbf{m} \text{ with } n \text{ vertices}\}\Big),$$

defining a probability measure. Uniform on $2p$-angulations if $q_k = \delta_{kp}$. Note $W_q^n = W_{q'}^n$ if $q'_k = \beta^{k-1} q_k$.

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Scaling limits of random $2p$-angulations

- Let M_n be a uniform $2p$-angulation with n vertices. Endow the set $V(M_n)$ of its vertices with the usual graph distance d_{gr}. Then ([Chassaing-Schaeffer], for $p = 2$) it holds that typical distances are of order $n^{1/4}$ as $n \to \infty$.
- More generally, one expects a convergence of the form

$$(V(M_n), n^{-1/4} d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} (S, c_p d)\,, \qquad (1)$$

for some constant $c_p > 0$, where (S, d) is a random metric space, the Brownian map.

Theorem (Le Gall, Le Gall-Paulin)

For every incresing sequence in $\mathbb{N}$, there exists a sub-sequence along which the convergence (1) holds in distribution for the Gromov-Hausdorff topology on compact metric spaces. The limit (S, d) is a.s. homeomorphic to $\mathbb{S}_2$, and has Hausdorff dimension a.s.

$$\dim_{\mathcal{H}}(S, d) = 4\,.$$

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Gromov-Hausdorff distance

A natural framework to manipulate random metric spaces is to compare two such spaces using the Gromov-Hausdorff distance . Let $(X,d),(X',d')$ be compact metric spaces, and set

$$\mathrm{d_{GH}}(X,X') = \inf_{\phi,\phi'} \delta_H(\phi(X),\phi'(X')) ,$$

the infimum being taken over all isometric embeddings of X, X' into a common metric space (Z,δ), and

$$\delta_H(K,K') = \max\left(\sup_{x\in K}\delta(x,K'), \sup_{x'\in K'}\delta(x',K)\right)$$

is the Hausdorff distance between compact sets in Z.

Proposition

The function $\mathrm{d_{GH}}$ is a complete and separable metric on the set of isometry classes of compact metric spaces.

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Scaling limits for Boltzmann-distributed maps

- When q is 'regular enough' (e.g. decreasing sufficiently fast), then degrees of typical faces of a W_q^n-sampled map are exponentially tight. Intuitively, faces remain small and distances are still of the order $n^{1/4}$. One expects the scaling limit to be still the Brownian map [Marckert-Miermont,Miermont-Weill].
- But if for some $a \in (3/2, 5/2)$, we have

$$q_k^\circ \sim k^{-a}, \qquad k \to \infty$$

and $q_k = c\, q_k^\circ$ (or $c\beta^{k-1} q_k^\circ$) for the appropriate "critical" value $c > 0$, then the largest face in a W_q^n-sampled map has degree of order $n^{1/\alpha}$.
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Main result

- We assume $q_k = c\, q_k^\circ$ where $q_k^\circ \sim k^{-a}$ for some $a \in (3/2, 5/2)$, $c > 0$ a critical value (explicit in terms of q°).
- Let $\alpha = a - 1/2 \in (1, 2)$.
- Let M_n be a map with distribution W_q^n.

Theorem

For every increasing sequence, there exists a subsequence along which

$$(V(M_n), n^{-1/2\alpha} d_{\mathrm{gr}}) \xrightarrow[n\to\infty]{(d)} (\mathbf{M}_\infty, \delta_\infty),$$

for the Gromov-Hausdorff topology. Moreover, the limiting space $(\mathbf{M}_\infty, \delta_\infty)$ *has Hausdorff dimension* $\dim_{\mathcal{H}}(\mathbf{M}_\infty, \delta_\infty) = 2\alpha$ *a.s.*

- The limit is not the Brownian map, we have a one-parameter family of pairwise distinct limit spaces.
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More on the limit space

- The distribution of the limit has not been characterized so far. This would allow to get rid of the "subsequence" part of the theorem.
- However, more is known about distances. More precisely, let v_* be uniformly chosen at random in M_n. Then we are able to describe all distances of the form $d_{\mathrm{gr}}(v_*, v)$, for $v \in V(M_n)$.
- Using a certain exploration of $V(M_n)$ and passing to the limit, we describe all distances from a randomly chosen point in $\mathbf{M}_\infty$ *via* a continuous distance function $(D_t, 0 \leq t \leq 1)$ (not a Markov process).
- This allows e.g. to describe the scaling limit of the profile of M_n seen from v_*:

$$\int \rho_{M_n}^{(n)}(\mathrm{d}x)\varphi(x) = \frac{1}{n} \sum_{v \in V(M_n)} \varphi(n^{-1/2\alpha} d_{\mathrm{gr}}(v_*, v)),$$

which converges to a simple functional of D.

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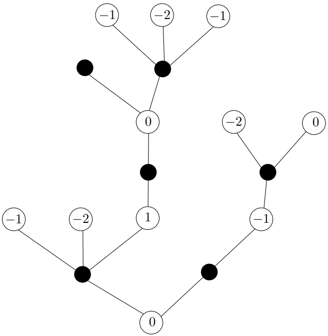
Main tool: bijective methods

- A now standard tool to attack scaling limits problems on random maps is to use bijective encodings of maps by tree structures whose scaling limit is easier to determine.
- We use the Bouttier-Di Francesco-Guitter (BDG) bijection between rooted, pointed bipartite maps and mobiles.



A mobile is a pair $(\mathcal{T}, (\ell(v)_{v\in\mathcal{T}^\circ}))$ where

- $\mathcal{T}$ a rooted plane tree: vertices $\mathcal{T}^\circ$ at even generations are white, others are black $\mathcal{T}^\bullet$.
- $\ell : \mathcal{T}^\circ \to \mathbb{Z}$ is a label function with $\ell(\text{root}) = 0$ and

$$\ell(v_{(i+1)}) - \ell(v_{(i)}) \geq -1,$$

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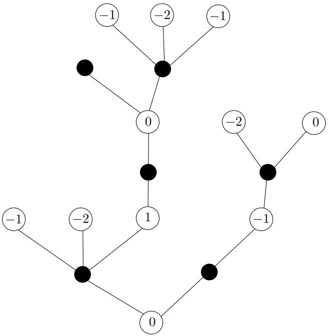

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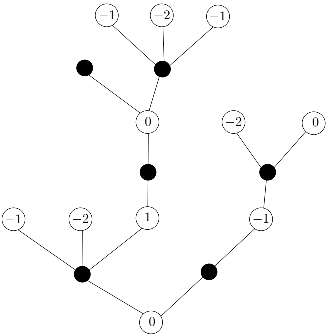

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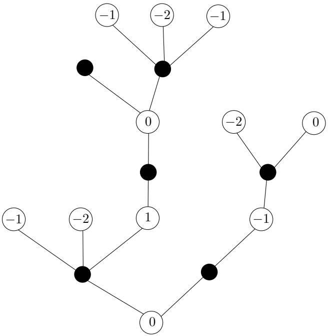

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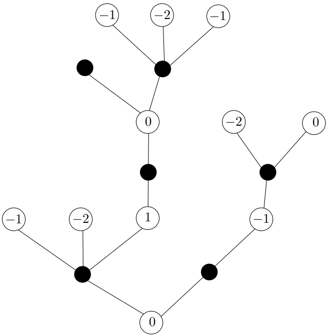

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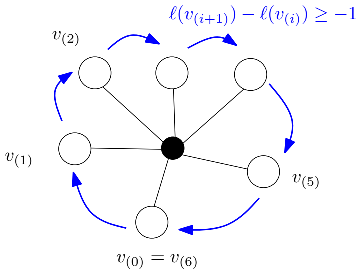
- A now standard tool to attack scaling limits problems on random maps is to use bijective encodings of maps by tree structures whose scaling limit is easier to determine.
- We use the Bouttier-Di Francesco-Guitter (BDG) bijection between rooted, pointed bipartite maps and mobiles.



Discrete bridge

A mobile is a pair $(\mathcal{T}, (\ell(v)_{v \in \mathcal{T}^\circ}))$ where

- $\mathcal{T}$ a rooted plane tree: vertices $\mathcal{T}^\circ$ at even generations are white, others are black $\mathcal{T}^\bullet$.
- $\ell : \mathcal{T}^\circ \to \mathbb{Z}$ is a label function with $\ell(\text{root}) = 0$ and

$$\ell(v_{(i+1)}) - \ell(v_{(i)}) \geq -1,$$

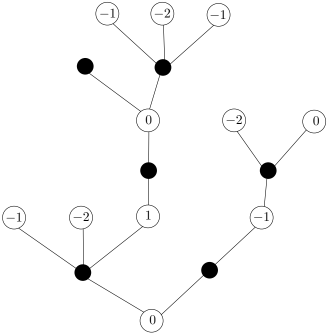
where $v_{(0)}, v_{(1)}, \ldots, v_{(k)}, v_{(k+1)} = v_{(0)}$ are the white vertices around a given black vertex, in clockwise order.

The BDG bijection

- Start from a mobile $\theta = (\mathcal{T}, \ell)$ with $n+1$ vertices.
- Let $v_0^\circ = \mathrm{root}, v_1^\circ, v_2^\circ, \ldots, v_{n-1}^\circ$ be the contour exploration of white vertices, extended by periodicity to $v_i^\circ, i \geq 0$.
- Add a vertex v_* not in $\mathcal{T}$, set $v_\infty^\circ = v_*$ by convention.
- For every $i \geq 0$, draw an edge between v_i° and $v_{\phi(i)}^\circ$ where

$$\phi(i) = \inf\{j \geq i : \ell(v_j^\circ) = \ell(v_i^\circ) - 1\} .$$

- Root the graph at the edge from $v_{\phi(0)}^\circ$ to v_0°.
- Remove edges incident to $\mathcal{T}^\bullet$.



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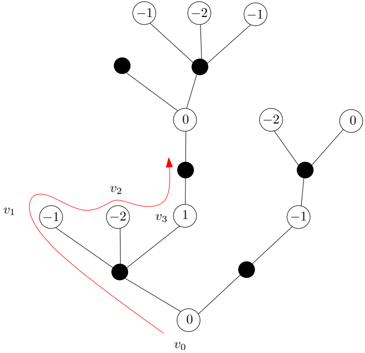

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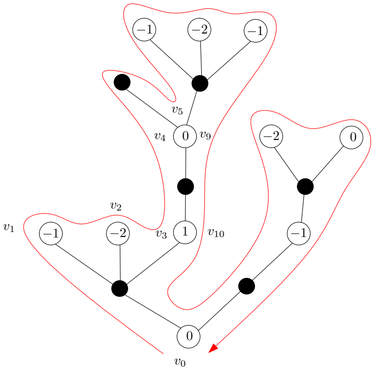

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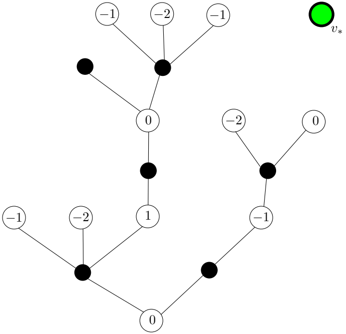

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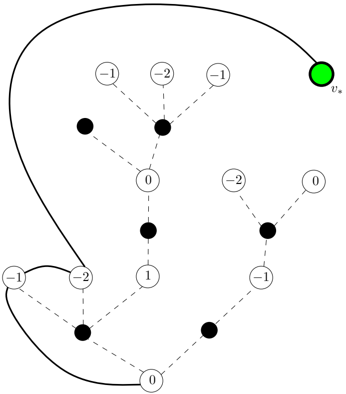

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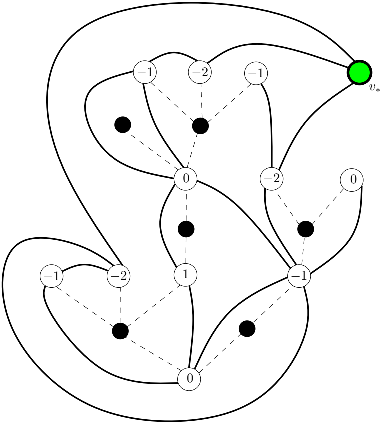

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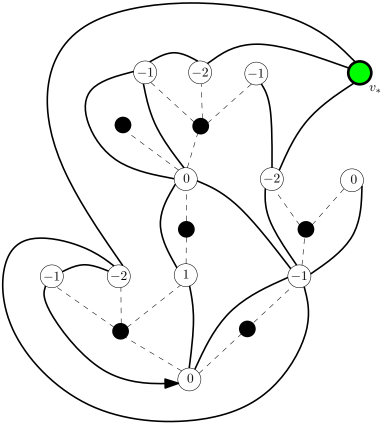

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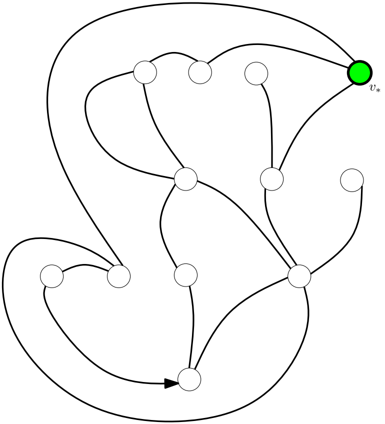
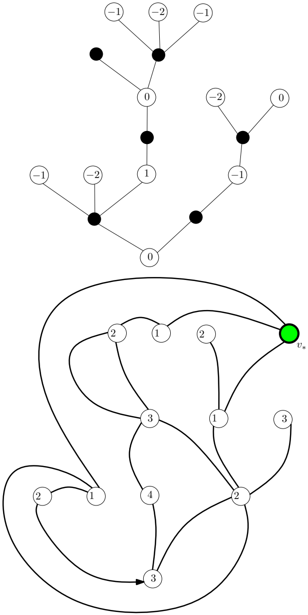


Properties of the BDG bijection



Proposition

This yields a bijection between

1. *Mobiles $\theta = (\mathcal{T}, \ell)$, and*
2. *bipartite, rooted and pointed maps $(\mathbf{m}, v_*, e)$ such that (positivity)*

$$d_{\mathrm{gr}}(v_*, e_-) = d_{\mathrm{gr}}(v_*, e_+) - 1\,.$$

- *A vertex $v \in \mathcal{T}^\circ$ corresponds to a vertex $v \in V(\mathbf{m}) \setminus \{v_*\}$ such that*

$$d_{\mathrm{gr}}(v, v_*) = \ell(v) - \min \ell + 1$$

- *A vertex $v \in \mathcal{T}^\bullet$ with k children corresponds to a face of $\mathbf{m}$ of degree $2k + 2$.*

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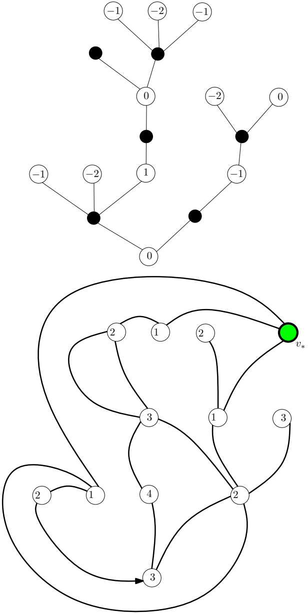

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Boltzmann distributions and the BDG bijection

Assume (M, v_*, e) has a Boltzmann distribution

$$P_q(\mathbf{m}, v_*, e) = Z_q^{-1} \prod_{f \in F(\mathbf{m})} q_{\deg(f)/2} \,.$$

Let $\theta = (\mathcal{T}, \ell)$ be the random mobile associated with M.

Proposition

- *The tree $\mathcal{T}$ is a Galton-Watson tree with two alternating types, and respective (white, black) offspring distributions* $\mu_0(k) = Z_q^{-1}(1 - Z_q^{-1})^k, k \geq 0$, *and*

$$\mu_1(k) = \frac{Z_q^k \binom{2k+1}{k} q_{k+1}}{f_q(Z_q)}, \qquad k \geq 0.$$

- *Conditionally on $\mathcal{T}$, the labels ℓ are uniform among labels satisfying the constraints in the definition of mobiles.*

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Large faces in scaling limits

- For an appropriate choice of q_k, in the form

$$q_k = c\beta^k q_k^\circ\,, \qquad q_k^\circ \sim k^a\,,$$

  $a \in (3/2, 5/2)$, the tree $\mathcal{T}$ is critical and

$$\mu_1([k,\infty)) \sim C_q k^{-\alpha}\,, \qquad k \to \infty\,,$$

  $\alpha = a - 1/2$. This says that the degree of a typical face of M (the offspring distribution of a typical vertex of $\mathcal{T}^\bullet$) is in the domain of attraction of a stable(α) random variable.
- Conditioning on the number of vertices of M to be $n+1$ (n the number of vertices of $\mathcal{T}^\circ$), the largest faces will have degrees of order $n^{1/\alpha}$ and follow a Poissonian-like repartition.

Discrete and continuous distance process

Let M_n have distribution W_q^n, v_* a uniformly chosen vertex in M_n, $\theta_n = (\mathcal{T}_n, \ell_n)$ the associated mobile, $v_0^\circ, v_1^\circ, \ldots$ the contour sequence.

$$\Lambda_i^{\theta_n} = \ell_n(v_i^\circ) \,, \qquad i \geq 0$$

(0 for $i \geq \#\mathcal{T}$). Recall that ℓ_n measures distances in M_n:

$$d_{\mathrm{gr}}(v_i^\circ, v_*) = \ell_n(v_i^\circ) - \min \ell_n + 1 = \Lambda_i^{\theta_n} - \underline{\Lambda}^{\theta_n} + 1 \,.$$

Proposition

As $n \to \infty$, we have the following convergence in distribution in the Skorokhod space:

$$\left(n^{-1/2\alpha} \Lambda_{[nt]}^{\theta_n}, t \geq 0\right) \xrightarrow[n\to\infty]{(d)} (D_t, t \geq 0) \,,$$

where $(D_t, t \geq 0)$ is a continuous stochastic process called the continuous distance process.

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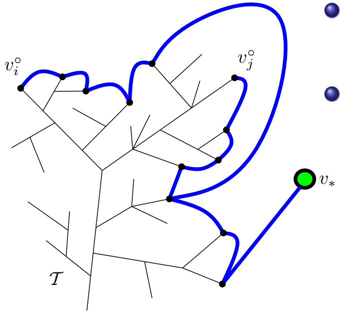
Gromov-Hausdorff tightness of W_q^n-distributed maps

The previous proposition is sufficient to prove tightness of the spaces $(V(M_n), n^{-1/2\alpha} d_{\mathrm{gr}}), n \geq 1$, through the following control on distances

$$d_{\mathrm{gr}}(v_i^\circ, v_j^\circ) \leq \Lambda_i^{\theta_n} + \Lambda_j^{\theta_n} - 2 \min_{i \wedge j \leq k \leq i \vee j} \Lambda_k^{\theta_n} + 2 .$$

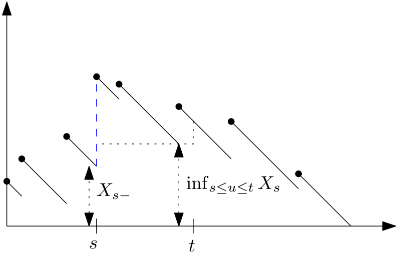
Blue: arches vertex - successor



- Indeed, the last successor of v_i° between v_i°, v_j° has label $\min_{i \leq k \leq j} \Lambda_k^{\theta_n}$
- Combined with the previous convergence, and the fact that the limiting process D is continuous, this is enough to get uniform bounds on the minimal number of balls of fixed radius needed to cover M_n.

Description of the continuous distance process

- Let $(X_t, 0 \le t \le 1)$ be the standard excursion above its minimum of a stable(α) Lévy process with only positive jumps.



A simplifying picture (making as if X were of finite variation)

- With each jump of X, say s such that $\Delta X_s = X_s - X_{s-} > 0$, associate an independent Brownian bridge

$$(b_s(u), 0 \le u \le \Delta X_s)$$

with duration ΔX_t.

- Set

$$D_t = \sum_{0<s\le t} b_s\left(\left(\inf_{s\le u\le t} X_u - X_{s-}\right)^+\right)$$

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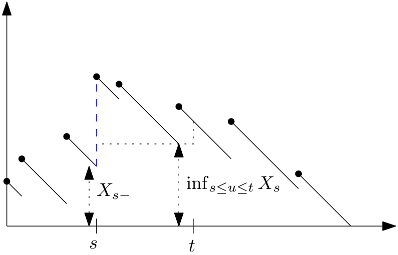

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A short explanation of the formula

- To show that D is the scaling limit of Λ^{θ_n}, must obtain an analogous formula in the discrete setting
- This is achieved using classical encodings of plane trees by random walks.
- For each v, $\ell_n(v)$ is a sum of discrete random bridges, one for each vertex being an ancestor in $\mathcal{T}^\bullet$ of v, and translate this in terms of the encoding random walks.
- Typically, each random discrete bridge has length of order $n^{1/\alpha}$, so they contribute approximately $n^{1/2\alpha}$ times a Brownian bridge to the label by Donsker's invariance principle for bridge.
- The terms $b_s\left((\inf_{s\leq u\leq t} X_u - X_{s-})^+\right)$ come from the fact that the bridges must be evaluated at times that correspond to the rank among its brothers, of the one vertex that is an ancestor of v.

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A motivation from physics: $O(N)$ models

Let **q** be a rooted quadrangulation, i.e. a rooted (planar) map with faces all of degree 4.

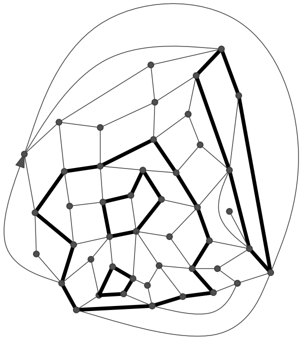
A loop configuration on **q** is a collection $\mathcal{L} = \{c_1, \ldots, c_k\}$, where

- $c_1, \ldots, c_k$ are simple cycles,
- the c_i's are non-intersecting

Set

$$\#\mathcal{L} = k \quad \text{and} \quad \lg(\mathcal{L}) = \sum_{i=1}^{k} \lg(c_i) ,$$

where $\lg(c_i)$ is the number of edges in the path c_i.



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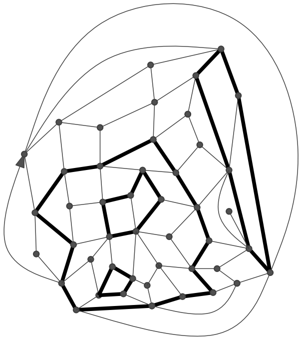


O(N) measure on random quadrangulations

- Let $N \geq 0$ be fixed. Let $\beta, x > 0$ be positive numbers.
- On the set of pairs $(\mathbf{q}, \mathcal{L})$, where
  - $\mathbf{q}$ is a rooted quadrangulation
  - $\mathcal{L}$ is a loop configuration on $\mathbf{q}$,

  we define a σ-finite measure by

$$W_{O(N)}(\mathbf{q}, \mathcal{L}) = e^{-\beta \# F(\mathbf{q})} x^{\lg(\mathcal{L})} N^{\# \mathcal{L}} ,$$

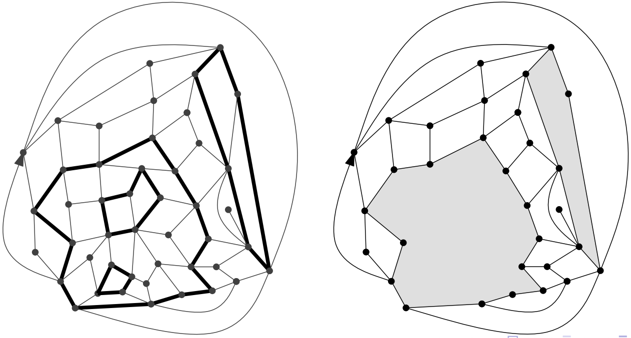
  the annealed $O(N)$ measure on random quadrangulations.
- When the total mass is finite:

$$Z_{O(N)}(\beta, x) = W_{O(N)}(1) < \infty ,$$

  we define a probability measure $P_{O(N)}$ by renormalizing $W_{O(N)}$ by $Z_{O(N)}(\beta, x)$.

Exterior gaskets

- Let $(\mathbf{q}, \mathcal{L})$ be a configuration. A cycle $c \in \mathcal{L}$ has an interior (the component of $\mathbb{S}_2 \setminus c$ not containing the face incident to the root)
- Deleting the interior of all cycles $c \in \mathcal{L}$, get the external gasket of $\mathcal{E}(\mathbf{q}, \mathcal{L})$.
- The map $\mathcal{E}(\mathbf{q}, \mathcal{L})$ has two types of faces: native quadrangles $Q(\mathbf{m})$ and holes $H(\mathbf{m})$ of any degree (shaded), with simple and mutually avoiding boundaries.



Boltzmann laws induced by $O(N)$ measures

- The law of the exterior gasket of an $O(N)$-model on quadrangulations is

$$W_{O(N)}(\{\mathcal{E}(\mathbf{q},\mathcal{L})=\mathbf{m}\}) = e^{-\beta\#Q(\mathbf{m})}\prod_{f\in H(\mathbf{m})} q_{\deg f/2}\,,$$

  where

$$q_k = x^{2k} Z^{\partial}_{O(N),k}(\beta,x)\,,$$

  where $Z^{\partial}_{O(N),k}(\beta,x)$ is the partition function for the $O(N)$-model with a boundary of length $2k$.
- This can be seen as a kind of Boltzmann distribution on random maps of the kind studied before.

Prediction from Physics

- Expect (see e.g. surveys by Duplantier), for $N = 2\cos(\pi\theta)$ with $\theta \in (0, 1/2)$, that there exists $x_c(\beta)$ a positive function, and $\beta_c > 0$ such that
  - for given $\beta > \beta_c, x = x_c(\beta)$, then as $k \to \infty$

    $$Z^{\partial}_{O(N),k}(\beta, x) \approx k^{-2+\theta}$$

  - for $\beta = \beta_c, x = x_c(\beta_c)$, then as $k \to \infty$

    $$Z^{\partial}_{O(N),k}(\beta, x) \approx k^{-2-\theta}$$

  respectively called dense and dilute phases.
- This should correspond to our models with $\alpha \in \{3/2 - \theta, 3/2 + \theta\}$. Note the conjectured coexistence when $\theta = 0, N = 2$.
- This should be related to conformal loop ensembles (Sheffield and Werner), and the KPZ formula linking models on random maps and regular lattices.

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The case of the Ising model

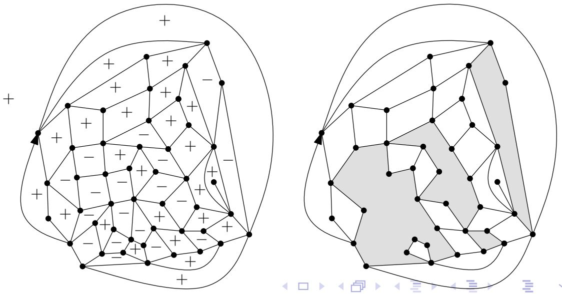
An Ising configuration is now a pair $(\mathbf{q}, \sigma)$ where $\mathbf{q}$ is a rooted quadrangulation, and

$$\sigma = (\sigma_f, f \in F(\mathbf{q})) \in \{-1, +1\}^{F(\mathbf{q})}$$

The (annealed) Ising measure is (J a real parameter)

$$W_I(\mathbf{q}, \sigma) = e^{-\beta \# F(\mathbf{q})} \exp\left(J \sum_{f \sim f'} \sigma_f \sigma_{f'} \right) ,$$

and define exterior gaskets in a similar fashion as for $O(N)$ models — Note that this time, the boundaries are only weakly avoiding



Predictions

- Predictions from physics [Kazakov] identify $J_c = \ln 2$ as critical
- Expects that respectively for $J = J_c$ or $J < J_c$ (and the appropriate values of β), the Ising model has the same scaling limit as the dilute and dense phases of the $O(N = 1)$ model
- These correspond to $\theta = 1/3$ and $\alpha \in \{11/6, 7/6\}$. Need to compute generating functions for Ising model with boundary.

Open problems and perspectives

1. Uniqueness of the limit laws.
2. Equivalent question: joint laws of mutual distances between k randomly sampled points.
3. Other geometric aspects of the limit.
4. Adding topological constraints on faces (self and mutually avoiding).