The mean perimeter of the convex hull of some plane random sets generated by the Brownian motion

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If C is a compact plane convex set, denote by $L(C)$ the length of its perimeter. Let $B(t) = (X(t), Y(t))$ be a standard Brownian motion in the plane and denote by $C_1(t)$ the convex hull of the curve

$$\mathbf{B} = \{B(s) \ ; 0 \leq s \leq t\}.$$

This object has been considered by Paul Lévy (1948), pages 239-240. The mean of $L(C_1(t))$ is $\sqrt{8\pi t}$. Here we consider the symmetric convex hull $C_2(t)$, namely the convex hull of $C_1(t) \cup -C_1(t)$.

By scaling we can guess the existence of a constant ℓ_2 such that $E[L(C_2(t))] = \ell_2\sqrt{8\pi t}$.

Similarly we consider random convex sets obtained in the following way. We let $\Omega \subset [0, 2\pi)$ be a set of angles and C_Ω the convex hull of $\cup_{\omega\in\Omega} R_\omega \mathbf{B}$, where R_ω is the rotation of angle ω. Again by scaling there exists a constant ℓ_Ω such that

$$E[L(C_\Omega(t))] = \ell_\Omega \sqrt{8\pi t}.$$

We shall compute this constant when Ω is equal to one of the following sets

$$\{0\}, \quad \{0, \pi\},$$

$$\{0, \pi/2\}, \quad \{0, \pi/2, \pi\}, \quad \{0, \pi/2, \pi, 3\pi/2\},$$

$$[0, 2\pi[, \quad \{0, 2\pi/3\}, \quad \{0, 2\pi/3, 4\pi/3\}.$$

giving the result either in closed form or as an expression which can be evaluated numerically.

Preliminary remarks Given a compact convex set C in the Euclidean plane $\mathbb{R}^2$ consider its support function h_C as the function on $\mathbb{R}$ of period 2π defined by

$$h_C(\theta) = \max_{(x,y)\in C} (x\cos\theta + y\sin\theta)$$

It is well known that

$$L(C) = \int_0^{2\pi} h_C(\theta)d\theta.$$

Let $\Omega \subset [0, 2\pi)$ and define

$$h_{\mathbf{B}}(\theta) = \max_{0\le s\le t} (X(s)\cos\theta + Y(s)\sin\theta)$$

and

$$h_\Omega(\theta) = \max_{\omega\in\Omega} h_{\mathbf{B}}(\theta+\omega).$$

The rotation invariance of the law of Brownian motion implies

$$E[L(C_\Omega(t))] = 2\pi E[h_\Omega(0)].$$

Let P_Ω be the convex set defined by the inequalities

$$x\cos\omega + y\sin\omega \le 1; \omega \in \Omega$$

and T_Ω be the first time Brownian motion exits this set, then a scaling argument shows that $h_\Omega(0)$ is distributed as $T_\Omega^{-1/2}$. It turns out that in several cases the Laplace transform $E[\exp(-\frac{\lambda^2}{2}T_\Omega)]$ is given by a simple formula, then

$$E[h_\Omega(0)] = E[T_\Omega^{-1/2}] = \sqrt{\frac{2}{\pi}}\int_0^\infty E[\exp(-\frac{\lambda^2}{2}T_\Omega)]d\lambda$$

and

$$E[L(C_\Omega)(t)] = \sqrt{8\pi t}\int_0^\infty E[\exp(-\frac{\lambda^2}{2}T_\Omega)]d\lambda = \sqrt{8\pi t}\ell_\Omega.$$

Thus

$$\int_0^\infty E[\exp(-\frac{\lambda^2}{2}T_\Omega)]d\lambda = \ell_\Omega.$$

Some easy cases : $\Omega = \{0\}$ The time T_Ω is the first hitting time of 1 for a linear Brownian motion, and

$$E[\exp(-\frac{\lambda^2}{2} T_\Omega)] = e^{-\lambda}. \tag{1}$$

The Laplace transform can be inverted to give

$$P(T_\Omega \in dt) = \frac{e^{-1/2t}}{\sqrt{2\pi t^3}}.dt \tag{2}$$

We get $E[L(C_1(t))] = \sqrt{8\pi t}$. As expected :

$$\ell_\Omega = 1.$$

$\Omega = \{0, \pi\}$ Now T_Ω is the exit time of linear Brownian motion from the interval $[-1, +1]$ and

$$E[\exp(-\frac{\lambda^2}{2} T_\Omega)] = 1/\cosh\lambda \qquad (3)$$

with density

$$P(T_\Omega \in dt) = \pi \sum_{n=0}^{\infty} (-1)^n (n+\frac{1}{2}) e^{-(n+\frac{1}{2})^2 \pi^2 t/2} dt \qquad (4)$$

and distribution function

$$P(T_\Omega \geq t) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\frac{1}{2}} e^{-(n+\frac{1}{2})^2 \pi^2 t/2}$$

see e.g. Biane, Pitman, Yor (2001) or Feller (1966). Thus

$$\ell_\Omega = \frac{\pi}{2}.$$

$\Omega = \{0, \pi/2\}$

We have $h_\Omega(\theta) = \max_{0 \le s \le t}$

$\{X(s)\cos\theta + Y(s)\sin\theta,\ Y(s)\cos\theta - X(s)\sin\theta\ \}$.

The trick is to use the fact that $(X(s)\cos\theta + Y(s)\sin\theta)_{s\ge 0}$ and $(Y(s)\cos\theta - X(s)\sin\theta)_{s\ge 0}$ are two standard one dimensional Brownian motions which are independent. As a consequence, using (2) we have

$$\Pr(h_\Omega(\theta) \le h) = H^2(h/\sqrt{t})$$

where

$$H(z) = -1 + 2\int_{-\infty}^{z} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}. \qquad (5)$$

It leads to $\mathbb{E}(L(\Omega) =$

$$2\pi \int_0^\infty (1 - H^2(h/\sqrt{t}))dh = 2\pi\sqrt{t}\int_0^\infty (1 - H^2(z))dz.$$

We write

$$1 - H^2(z) = 4\int_z^\infty e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \left(1 - \int_z^\infty e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}\right)$$

By the change of variables $x = zu$ and $y = zv$ and $w = z^2/2$ we get easily

$$\mathbb{E}(L(C_\Omega)) = \sqrt{8\pi t}\left(2 - \int_1^\infty \int_1^\infty \frac{dudv}{(u^2 + v^2)^{3/2}}\right)$$

Introduce the function $(x, y) \mapsto g(x, y)$ defined for x and y in $\mathbb{R} \setminus \{0\}$ by

$$g(x, y) = \frac{\sqrt{x^2 + y^2}}{xy}. \qquad (6)$$

A tedious calculation shows that if $a < b$ and $c < d$ with $(0,0) \notin [a,b] \times [c,d]$ and if furthermore $a,b,c,d \neq 0$ we have

$$\int_a^b \int_c^d \frac{dudv}{(u^2+v^2)^{3/2}} = g(a,c)+g(b,d)-g(a,d)-g(b,c). \tag{7}$$

From this we get that

$$\int_1^\infty \int_1^\infty \frac{dudv}{(u^2+v^2)^{3/2}} = 2-\sqrt{2}$$

which finally gives that

$$\ell_\Omega = \sqrt{2}.$$

$\Omega = \{0, \pi/2, \pi, 3\pi/2\}$ The method is quite similar to the method used before, but the distribution function H of the maximum of a Brownian motion is replaced by the distribution function L of the maximum of its absolute value. Still with the previous notations we have

$$h_{C_\Omega}(\theta) = \max\{|X(s)\cos\theta + Y(s)\sin\theta|,$$

$$|Y(s)\cos\theta - X(s)\sin\theta| \ ; \ 0 \leq s \leq t\}.$$

We use again the fact that $(X(s)\cos\theta + Y(s)\sin\theta)_{s\geq 0}$ and $(Y(s)\cos\theta - X(s)\sin\theta)_{s\geq 0}$ are two standard one dimensional independent Brownian motions. We get

$$\Pr(h_{C_\Omega}(\theta) \leq h) = L^2(h/\sqrt{t})$$

and $\mathbb{E}(L(C_\Omega)) =$

$$\int_0^{2\pi} \mathbb{E}(h_{C_4(t)}(\theta))d\theta = 2\pi \int_0^{\infty} (1 - L^2(h/\sqrt{t}))dh =$$

$$2\pi\sqrt{t} \int_0^{\infty} (1 - L^2(z))dz.$$

We now write

$$1 - L^2(z) = (1 - L(z))(1 + L(z)) =$$

$$\frac{2}{\pi} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{(4k-1)z}^{(4k+1)z} e^{-\frac{x^2}{2}} dx \int_{(4n+1)z}^{(4n+3)z} e^{-\frac{y^2}{2}} dy.$$

We are led to the computation of the integral for fixed $(k, n) \in \mathbb{Z}^2$

$$I(k,n) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\int_{(4k-1)z}^{(4k+1)z} e^{-\frac{x^2}{2}} dx \int_{(4n+1)z}^{(4n+3)z} e^{-\frac{y^2}{2}} dy \right) dz.$$

By the same change of variables as before $x = zu$ and $y = zv$ and $w = z^2/2$ we get

$$I(k,n) = \int_{4k-1}^{4k+1} \int_{4n+1}^{4n+3} \frac{du dv}{(u^2 + v^2)^{3/2}}. \qquad (8)$$

In order to compute $I(k, n)$ we use the function g defined by (6) and the identity (7).

$$\ell_\Omega = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I(k, n) \tag{9}$$

One can observe that we have the following two symmetries : for all $(k, n) \in \mathbb{Z}^2$

$$I(k, n) = I(-k, n) = I(k, -n - 1).$$

$\ell_\Omega = S_0 + S_1$ with

$$S_0 = \sum_{n=-\infty}^{\infty} I(0,n) =$$

$$4 \sum_{n=0}^{\infty} \left(\frac{\sqrt{1+(4n+1)^2}}{4n+1} - \frac{\sqrt{1+(4n+3)^2}}{4n+3} \right)$$

and

$$S_1 = 4 \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} I(k,n).$$

We compute $\frac{S_0}{4} \approx 0.3725...$ and

$$\ell_\Omega = 1,9178...$$

by Mathematica.

$\Omega = \{0, \pi/2, \pi\}$ Again the same method gives

$$\mathbb{E}(L(C_\Omega)) = 2\pi \int_0^\infty (1 - L(h/\sqrt{t})H(h/\sqrt{t}))dh =$$

$$2\pi\sqrt{t} \int_0^\infty (1 - L(z)H(z))dz.$$

We write for simplicity $L = 1 - 2L_1$ and $H = 1 - 2H_1$ so that

$$\mathbb{E}(L(C_\Omega)) =$$

$$2\pi\sqrt{t} \int_0^\infty (2L_1(z) + 2H_1(z) - 4L_1(z)H_1(z))\, dz.$$

A previous calculation has shown that

$$2\pi\sqrt{t} \int_0^\infty 2L_1(z)dz = \frac{\pi}{2}\sqrt{8\pi t}$$

and a direct calculation shows 2π and $y = zv$ and $w = z^2/2$, the use of the function g of (6) and the formula (7) we get

$$2\pi\sqrt{t}\int_0^\infty 4L_1(z)H_1(z)dz =$$

$$\sqrt{8\pi t}\sum_{n=-\infty}^{\infty}\int_1^\infty\int_{4n+1}^{4n+3}\frac{dudv}{(u^2+v^2)^{3/2}} =$$

$$\sqrt{8\pi t}\sum_{n=-\infty}^{\infty}(g(1,4n+1)-g(1,4n+3)).$$

It is easily seen that

$$\sum_{n=-\infty}^{\infty}(g(1,4n+1)-g(1,4n+3)) = \frac{S_0}{2} \approx 0.7450... \tag{10}$$

Thus

$$\ell_\Omega \approx \frac{\pi}{2} + 0.2550....$$

$\Omega = [0, 2\pi)$ The set P_Ω is the unit circle, and the time T_Ω is the first hitting time of 1 by a Bessel process of dimension 2, therefore

$$\mathbb{E}(e^{-\frac{\lambda^2}{2}T_\Omega}) = \frac{I_0(0)}{I_0(\lambda)} \qquad (11)$$

where

$$I_0(2z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!^2}$$

is the Bessel function. Thus

$$\ell_\Omega = \int_0^\infty \frac{d\lambda}{I_0(\lambda)}.$$

A calculation by Mathematica gives

$$\ell_\Omega = 2,08323..$$

.

$\Omega = \{0, 2\pi/3\}$.

T_Ω is the exit time from a cone of angle $\pi/3$ tangent to the unit circle. In this case one can use results from Doumerc and O'Connell (2005). There, the exit time from the cone $x_1 > x_2 > x_3$ in $\mathbf{R}^3$, starting from the point (x_1, x_2, x_3) is expressed as

$$P(T \geq t) = p_{12} - p_{13} + p_{23}$$

with $p_{ij} = \sqrt{\frac{2}{\pi}} \int_0^{(x_i - x_j)/\sqrt{2t}} e^{-y^2/2} dy$.

Taking the component of Brownian motion in the hyperplane $x_1 + x_2 + x_3 = 0$, with the starting point $(\sqrt{2}, 0, -\sqrt{2})$, the trick is to observe that if $b = (b_1, b_2, b_3)$ is in $\mathbb{R}^3$ and if $x = (x_1, x_2, x_3)$ is its orthogonal projection on $x_1 + x_2 + x_3 = 0$, then $b \in C$ if and only if $x \in C$. One gets

$$P(T_\Omega \geq t) = \sqrt{\frac{2}{\pi}} \left(2 \int_0^{1/\sqrt{t}} e^{-y^2/2} dy - \int_0^{2/\sqrt{t}} e^{-y^2/2} dy \right)$$

and a straightforward computation gives

$$E[T_\Omega^{-1/2}] = \frac{3}{2}\sqrt{\frac{2}{\pi}}$$

and

$$E[L(C_\Omega(t))] = \frac{3}{2}\sqrt{8\pi t}.$$

Thus $\ell_\Omega = 3/2$.

$\Omega = \{0, 2\pi/3, 4\pi/3\}$ Hence P_Ω is an equilateral triangle centered at 0. We will compute the distribution of T_Ω and get a curious identity in law.

In order to do the computation it will be convenient to take the equilateral triangle Δ with vertices $0, 1, -j^2$ where $j = (-1+i\sqrt{3})/2$. Its center is $x_0 = (1-j^2)/3$. The case of the triangle P_Ω can be obtained by an affinity. The orthogonal reflections with respect to the three lines bordering the triangle Δ generate a group W of affine isometries of the Euclidian plane, whose fundamental domain is Δ. Using the reflection principle, we can compute the probability transition of Brownian motion in the triangle, killed at the boundary. One obtains

$$p_t^0(x,y) = \sum_{w \in W} \det(w) p_t(x, w(y))$$

where $p_t(x,y)$ is the ordinary heat kernel on the Euclidian plane and x and y are points of this plane.

If we denote R the lattice generated by 1 and j, then the dual lattice $\widehat{R}$ (such that $\langle w, \widehat{w} \rangle \in \mathbf{Z}$) is generated by the dual basis $a = 1 + \frac{i}{\sqrt{3}}$ and $b = \frac{2i}{\sqrt{3}}$. The plane is tiled by translates of Δ by R, which correspond to direct isometries $w \in W$ and translates of $\bar{\Delta}$. Thus

$$p_t^0(x, y) = \sum_{r \in R} p_t(x - y, r) - p_t(x - \bar{y}, r).$$

The function

$$h(t, z) = \sum_{r \in R} p_t(z, r)$$

is periodic in z with period lattice R and $p_t^0(x, y) = h(t, x - y) - h(t, x - \bar{y})$. One can apply Poisson summation formula, i.e. expand it in terms of the characters $e^{2i\pi\langle ., \widehat{w} \rangle}$ obtaining

$$h(t, z) = \frac{2}{\sqrt{3}} \sum_{\widehat{r} \in \widehat{R}} e^{2i\pi\langle z, \widehat{r} \rangle} e^{-2\pi^2 |\widehat{r}|^2 t}$$

Let us now integrate with respect to y in the triangle to obtain the distribution function for T, the first exit time of Brownian motion starting from x_0,

$$P(T > t) = \int_\Delta p_t^0(x_0, y)dy.$$

We use the coordinates $y = y_1 + y_2 j, 1 \geq y_1 \geq y_2 \geq 0$ on the triangle, with Jacobian $\frac{\sqrt{3}}{2}$ and $\hat{r} = ma + nb$ on $\hat{R}$, then

$$\langle y, \hat{r} \rangle = my_1 + ny_2, \qquad \langle x_0, \hat{r} \rangle = 2m/3 + n/3,$$

$$|\hat{w}|^2 = 4(m^2 + mn + n^2)/3.$$

Furthermore, by periodicity

$$\int_\Delta h(t, x - \bar{y})dy = \int_\Delta h(t, x - j - \bar{y})dy = \int_{\bar{\Delta}+j} h(t, x - y)dy$$

and the triangle $\bar{\Delta} + j$ corresponds to the coordinates $1 \geq y_2 \geq y_1 \geq 0$.

Let us compute

$$\int_{\Delta} h(t, x_0 - y)dy = \frac{\sqrt{3}}{2} \sum_{n,m} e^{2i\pi(2m/3+n/3)} \times$$

$$\int_0^1 dy_1 \int_0^{y_1} e^{-2i\pi(my_1+ny_2)} dy_2 e^{-8\pi^2(m^2+mn+n^2)t/3}$$

It is easy to see that terms with $n \neq 0, m \neq 0, m + n \neq 0$ give 0 contribution. Also the term $m = n = 0$ will be cancelled by the other integral. The sequence $m = 0, n \neq 0$, gives

$$
\begin{aligned}
& \sum_{n \neq 0} e^{2i\pi n/3} \int_0^1 dy_1 \int_0^{y_1} e^{-2i\pi n y_2} dy_2 e^{-8\pi^2 n^2 t/3} \\
= & \sum_{n \neq 0} e^{2i\pi n/3} \int_0^1 \frac{1}{2i\pi n}(1 - e^{-2i\pi n y_1}) dy_1 e^{-8\pi^2 n^2 t/3} \\
= & \sum_{n \neq 0} e^{2i\pi n/3} \frac{1}{2i\pi n} e^{-8\pi^2 n^2 t/3}
\end{aligned}
$$

The sequence $m \neq 0, n = 0$, gives

$$\begin{aligned} & \sum_{m\neq 0} e^{4i\pi m/3} \int_0^1 dy_1 e^{-2i\pi m y_1} \int_0^{y_1} dy_2 e^{-8\pi^2 m^2 t/3} \\ = & \sum_{m\neq 0} e^{4i\pi m/3} \int_0^1 dy_1 y_1 e^{-2i\pi y_1 m} e^{-8\pi^2 m^2 t/3} \\ = & -\sum_{m\neq 0} e^{4i\pi m/3} \frac{1}{2i\pi m} e^{-8\pi^2 m^2 t/3} \end{aligned}$$

The sequence $m \neq 0, m+n=0$, gives

$$\begin{aligned}
& \sum_{m \neq 0} e^{2i\pi m/3} \int_0^1 dy_1 \int_0^{y_1} e^{2i\pi(-my_1+my_2)} dy_2 e^{-8\pi^2 m^2 t/3} \\
= & \sum_{m \neq 0} e^{2i\pi m/3} \frac{1}{2i\pi m} \int_0^1 dy_1 (1 - e^{-2i\pi y_1 m}) e^{-8\pi^2 m^2 t/3} \\
= & \sum_{m \neq 0} e^{2i\pi m/3} \frac{1}{2i\pi m} e^{-8\pi^2 m^2 t/3}
\end{aligned}$$

The sum of these three terms is

$$3\sum_{n=1}^{\infty}\frac{\sin(2\pi n/3)}{\pi n}e^{-8\pi^2 n^2 t/3}$$

The other integral is

$$\int_{\Delta} h(t, x_0 - \bar{y})dy = \sum_{n,m} e^{2i\pi(2m/3+n/3)}$$

$$\int_0^1 dy_2 \int_0^{y_2} e^{-2i\pi(my_1+ny_2)}dy_1 e^{-8\pi^2(m^2+mn+n^2)t/3}$$

Using

$$\int_0^1 dy_1 \int_0^1 e^{-2i\pi(my_1+ny_2)} dy_1 dy_2 = 0$$

if m or n is $\neq 0$ we see that, except for the term $m = n = 0$ the other terms are the opposite of what we computed, therefore after some rearrangement

$$P(T > t) = 3\sqrt{3} \sum_{n=1}^{\infty} \frac{\chi_3(n)}{\pi n} e^{-8\pi^2 n^2 t/3}$$

where

$$\chi_3(n) = \frac{2}{\sqrt{3}} \sin(2\pi n/3)$$

is the multiplicative Dirichlet character modulo 3. The density is

$$P(T \in dt) = 8\sqrt{3} \sum_{n=1}^{\infty} \pi n \chi_3(n) e^{-8\pi^2 n^2 t/3}$$

A curious identity in law Let S be the first hitting time of $3a$ by a three-dimensional Bessel process starting from a. The Laplace transform is

$$E[e^{-\lambda^2 S/2}] = \frac{3\sinh(\lambda a)}{\sinh(3\lambda a)} = 3\sum_{n=1}^{\infty} \chi_3(n) e^{-2n\lambda a}.$$

Inverting term by term gives the density

$$\sum_{n=0}^{\infty} 6an \frac{\chi_3(n) e^{-2n^2a^2/t}}{\sqrt{2\pi t^3}}$$

We can also use the alternative expression

$$\frac{3\sinh(\lambda a)}{\sinh(3\lambda a)} = \sum_{n=1}^{\infty} \frac{3\sqrt{3}\pi n \chi_3(n)}{\pi^2 n^2 + 9\lambda^2 a^2}$$

and invert term by term the Laplace transform to get

$$P(S \in dt) = \sum_{n=1}^{\infty} \frac{3\sqrt{3}\pi n \chi_3(n)}{18a^2} e^{-\pi^2 n^2 t/(18a^2)}$$

The agreement between these two expressions is an instance of the functional equation of theta series. Putting $8/3 = 1/(18a^2)$, or $a = \frac{1}{4\sqrt{3}}$ we see that the exit time T and the time S have the same distribution. Coming back to T_Ω, the triangle P_Ω has sides of size $2\sqrt{3}$ therefore

$$E[e^{-\frac{\lambda^2}{2}T_\Omega}] = \frac{3\sinh(\lambda/2)}{\sinh(3\lambda/2)}.$$

Using (**??**) we are reduced to an elementary integral, and the result is

$$\ell_\Omega = \frac{\pi}{\sqrt{3}}.$$

Areas

Calculations for the mean areas rely on the formula $A(C) = \int_0^{2\pi}(h_C(\theta)^2 - h_C'(\theta)^2)d\theta$, where $h_C'(\theta)$ denotes the derivative on the left, which can be shown to exist and ultimately of the distribution of $h_C'(\theta)$ when C is random. This problem was solved in the unpublished thesis of El Bachir (1983) for $C_1(t)$, where

$$\mathbb{E}(A(C_1(t)) = \pi t/2$$

is proved. It is unsolved for the other convex sets considered here, except for $C_\infty(t)$ the smallest circle, centered at 0, surrounding $C_1(t)$, indeed

$$\mathbb{E}(A(C_\infty(t)) = \pi t \int_0^\infty \frac{ds}{I_0(\sqrt{2s})} = \pi t \times 3.06883.$$

a little more than six times the quantity $\mathbb{E}(A(C_1(t))$.

Conclusion...

Ω	ℓ_Ω
$\{0\}$	1
$\{0, \pi/2\}$	$\sqrt{2}$
$\{0, 2\pi/3\}$	$3/2$
$\{0, \pi\}$	$\pi/2$
$\{0, 2\pi/3, 4\pi/3\}$	$\pi/\sqrt{3}$
$\{0, \pi/2, \pi\}$	1.8257...
$\{0, \pi/2, \pi, 3\pi/2\}$	1.9178...
$(0, 2\pi)$	2.08323...

Three tantalizing questions

Find ℓ_Ω for

1. $\Omega = \{0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3\}$ (the hexagon)
2. $\Omega = \{0, \pi/n, 2\pi/n, 3\pi/n, \ldots, (n-1)\pi/n\}$ (the regular polygon)
3. $\Omega = \{0, \pi - \epsilon\}$ (a small angle)