

The n -tuple laws

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¹Based on 'Exact and asymptotic n -tuple laws at first and last passage' by K., Pardo and Rivero, to appear in Annals of Applied Probability

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- If X is a two-sided jumping strictly stable process with index $\alpha \in (0, 2)$ and positivity constant $\rho = \mathbb{P}(X_t \geq 0) \in (0, 1)$ then:
For $x \in (0, b)$, $u \in [0, b - x]$, $v \in [u, b)$ and $y > 0$,

$$\begin{aligned} & \mathbb{P}_x(b - \overline{X}_{\tau_b^{+-}} \in du, b - X_{\tau_b^{+-}} \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1 - \rho))} \times \\ & \quad \frac{x^{\alpha(1-\rho)}(b - x - u)^{\alpha\rho - 1}(v - u)^{\alpha(1-\rho) - 1}(b - v)^{\alpha\rho}}{(b - u)^\alpha(y + v)^{\alpha + 1}} du dv dy. \end{aligned}$$

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- In this talk: address the previous bullet point by examining ' n -tuple laws' for Lévy processes and positive self-similar Markov processes.

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- Π will be the Lévy measure and for $x > 0$, $\bar{\Pi}^+(x) = \Pi(x, \infty)$.
- Standard theory allows us to construct a local time at zero, say L , for the strong Markov Process $\bar{X} - X$. Then defining $H_t = X_{L_t^{-1}}$ (with the formality $H_\infty := \infty$) gives us the ascending ladder height processes (L^{-1}, H) . The pair (L^{-1}, H) is a (killed) bivariate subordinator with potential measure denoted by

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- The ladder processes has (amongst other things) hidden information about the distribution of \overline{X}_t , τ_x^+ and

$$\overline{G}_t = \sup\{s < t : X_s = \overline{X}_s\}.$$

The quintuple law at first passage

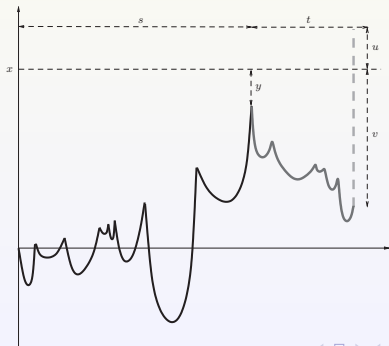
The quintuple law at first passage

■ Theorem (Doney and K. 2006)

For each $x > 0$ we have on $u > 0, v \geq y, y \in [0, x], s, t \geq 0$,

$$\begin{aligned} \mathbb{P}(\tau_x^+ - \overline{G}_{\tau_x^+} \in dt, \overline{G}_{\tau_x^+} \in ds, X_{\tau_x^+} - x \in du, x - X_{\tau_x^+} \in dv, x - \overline{X}_{\tau_x^+} \in dy) \\ = V(ds, x - dy) \widehat{V}(dt, dv - y) \Pi(du + v) \end{aligned}$$

where the equality holds up to a multiplicative constant.



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$$\mathbb{P}_x^\uparrow(X_t \in dz) = \frac{\widehat{V}(z)}{\widehat{V}(x)} \mathbb{P}_x(X_t \in z; \tau_0^- > t)$$

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- Tanaka-Doney construction of \mathbb{P}^\uparrow together with the quintuple law at first passage gives us a quintuple law at last passage.

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$$\underline{X}_{\rightarrow t} = \inf\{X_s : s \geq t\}$$

be the future infimum of X ,

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is the right end point of the excursion of X from its future infimum straddling time t . Now define the last passage time

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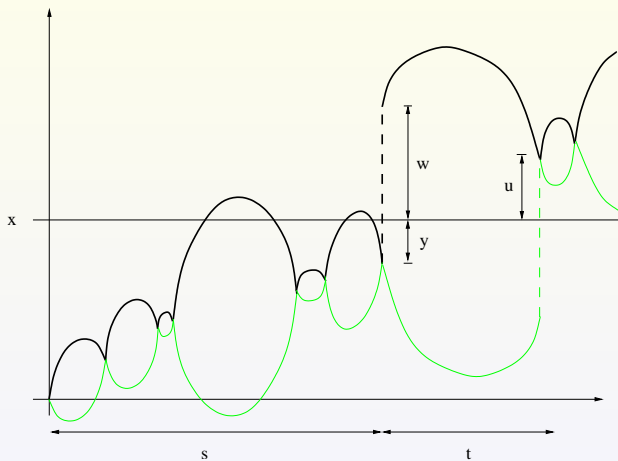
- Theorem**

Suppose that X is a Lévy process which does not drift to $-\infty$. For $s, t \geq 0$, $0 < y \leq x$, $w \geq u > 0$,

$$\begin{aligned} \mathbb{P}^\uparrow(\underline{D}_{\rightarrow U_x} - U_x \in dt, U_x \in ds, \underline{X}_{\rightarrow U_x} - x \in du, x - X_{U_x-} \in dy, X_{U_x} - x \in dw) \\ = V(ds, x - dy) \widehat{V}(dt, w - du) \Pi(dw + y) \end{aligned}$$

where the equality hold up to a multiplicative constant.

$$\{ \underline{D}_{U_x} - U_x \in dt, U_x \in ds, \underline{X}_{U_x} - x \in du, x - X_{U_x} \in dy, X_{U_x} - x \in dw \}$$



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■ Corollary

Suppose that X is a Lévy process which does not drift to $-\infty$. For $t > 0, x \geq z > 0, s > r > 0, 0 \leq v \leq z \wedge x, 0 < y \leq x - v, w \geq u > 0,$

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Playing with an idea of Lamperti, Caballero and Chaumont

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- The process $(X, \mathbb{P}_x^\uparrow)$ is a positive self-similar Markov process with index α meaning for $k > 0$, the law of $(kX_{k^{-\alpha}t}, t \geq 0)$ under \mathbb{P}_x^\uparrow is \mathbb{P}_{kx}^\uparrow and that it respects the Lamperti representation

$$X_t = x \exp\{\xi_{\theta(tx^{-\alpha})}\}$$

where $\theta(t) = \inf\{s \geq 0 : \int_0^s \exp\{\alpha X_u\} du > t\}$. and ξ is a Lévy process.

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- Using the law of the global infimum of a conditioned Lévy process applied to $(X, \mathbb{P}_x^\uparrow)$ one computes the law of the global infimum of ξ by the Lamperti-transformation and thereby obtains

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- One then uses Vigon's *equations amicales* to give us an expression for the jump measure of H : $\Pi_H(x, \infty) = \int_0^\infty \widehat{V}(du) \nu(u+x, \infty)$ from which it turns out to be easy to compute the potential

$$V(dx) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho)+1)} (1 - e^{-x})^{\alpha\rho-1} dx$$

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- The quintuple law at first passage for ξ marginalized to a triple law give us: For $y \in [0, x]$, $v \geq y$ and $u > 0$,

$$\begin{aligned} & \mathbb{P}(\xi_{\tau_x^+} - x \in du, x - \xi_{\tau_x^+} \in dv, x - \bar{\xi}_{\tau_x^+} \in dy) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} (1 - e^{-x+y})^{\alpha\rho-1} (1 - e^{-v+y})^{\alpha(1-\rho)-1} \\ & \quad \cdot e^{-v+y} e^{(\alpha(1-\rho)+1)(u+v)} (e^{u+v} - 1)^{-\alpha-1} dy dv du. \end{aligned}$$

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- Using the Lamperti representation this translates into a first passage problem for $(X, \mathbb{P}_x^\dagger)$. Let $b > x > 0$. For $u \in [0, b-x]$, $v \in [u, b)$ and $y > 0$,

$$\begin{aligned} & \mathbb{P}_x^\dagger(b - \bar{X}_{\tau_b^+} \in du, b - X_{\tau_b^+} \in dv, X_{\tau_b^+} - b \in dy) = \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \\ & \quad \times \frac{(b-x-u)^{\alpha\rho-1} (v-u)^{\alpha(1-\rho)-1} (b-v)^{\alpha\rho} (y+b)^{\alpha(1-\rho)}}{(b-u)^\alpha (y+v)^{\alpha+1}} du dv dy, \end{aligned}$$

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- Recalling $\mathbb{P}_x^\dagger(X_t \in dz) = (z/x)^{\alpha(1-\rho)} \mathbb{P}_x(X_t \in dz, t < \tau_0^-)$ we can use the last identity to deduce

$$\begin{aligned} \mathbb{P}_x(b - \overline{X}_{\tau_b^{+-}} \in du, b - X_{\tau_b^{+-}} \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-) &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \\ &\times \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^\alpha(y+v)^{\alpha+1}} du dv dy. \end{aligned}$$

Generating new identities playing X off against ξ

- Recalling $\mathbb{P}_x^\uparrow(X_t \in dz) = (z/x)^{\alpha(1-\rho)} \mathbb{P}_x(X_t \in dz, t < \tau_0^-)$ we can use the last identity to deduce

$$\begin{aligned} \mathbb{P}_x(b - \overline{X}_{\tau_b^{+-}} \in du, b - X_{\tau_b^{+-}} \in dv, X_{\tau_b^+} - b \in dy, \tau_b^+ < \tau_0^-) &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \\ &\times \frac{x^{\alpha(1-\rho)}(b-x-u)^{\alpha\rho-1}(v-u)^{\alpha(1-\rho)-1}(b-v)^{\alpha\rho}}{(b-u)^\alpha(y+v)^{\alpha+1}} du dv dy. \end{aligned}$$

- We can bring this identity back through the Doob h -transforms relating \mathbb{P}_x^\uparrow and \mathbb{P}_x and through the Lamperti transformation to give (with obvious notation): For $\theta \in [0, b]$, $\theta \leq \phi < b - u$ and $\eta > 0$

$$\begin{aligned} &\mathbb{P}\left(b - \overline{\xi}^\uparrow_{T_b^{+-}} \in d\theta, b - \xi^\uparrow_{T_b^{+-}} \in d\phi, \xi^\uparrow_{T_b^+} - b \in d\eta, T_b^+ < T_u^-\right) \\ &= \frac{\sin(\pi\alpha\rho)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} e^b (1 - e^u)^{\alpha(1-\rho)} e^{-\theta-\phi} e^{(\alpha(1-\rho)+1)\eta} (e^{b-\theta} - 1)^{\alpha\rho-1} (e^{-\theta} - e^{-\phi})^\alpha \\ &\quad \times (e^{b-\phi} - e^u)^{\alpha\rho} (e^{b-\theta} - e^u)^{-\alpha} (e^\eta - e^{-\phi})^{-\alpha-1} d\theta d\phi d\eta. \end{aligned}$$