Fractional fields indexed by metric spaces

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H-sssi fields indexed by $\mathbb{R}^n$

Fields invariant

- by isometry

$$(X(i(M)) - X(i(O)))_{M\in\mathbb{R}^n} \stackrel{\mathcal{D}}{=} (X(M) - X(O))_{M\in\mathbb{R}^n}$$

- by scaling $\lambda > 0$, index H

$$(X(\lambda M))_{M\in\mathbb{R}^n} \stackrel{\mathcal{D}}{=} \lambda^H (X(M))_{M\in\mathbb{R}^n}$$

Existance

- Gaussian case : $0 < H \leq 1$
- Stable case:
  - $0 < H \leq 1/\alpha$ if $0 < \alpha \leq 1$
  - $0 < H < 1$ if $1 < \alpha < 2$

Why this set of parameters (H, α)?

An application

Texture simulation: e.g. cloud

Texture brought by a surface?

H-sssi field

X H-sssi $\Rightarrow$ normalized increments constants in distribution

$$\frac{X(M) - X(N)}{\|MN\|^H} \quad \stackrel{\mathcal{D}}{=} \quad Ct$$

(equivalence if Gaussian)

Fields indexed by (E,d)

Replace $\|MN\|$ by $d(M,N)$:

$$\frac{X(M) - X(N)}{d(M,N)^H} \overset{\mathcal{D}}{=} Ct$$

First question: existance depending on H?

Distances of negative type (nt)

d is nt if

$$\forall n \geq 2;\ \forall \lambda_1, \ldots, \lambda_n, \sum_{i=1}^{n} \lambda_i = 0,\ \forall M_1, \ldots, M_n:$$

$$\sum_{i,j} \lambda_i \lambda_j d(M_i, M_j) \quad \leq \quad 0$$

Examples: Euclidean norm, geodesic distances on spheres and hyperbolic spaces and trees.

Consequences of Schönberg's theorem

$$\beta_E = \sup\{\beta > 0 \text{ st } d^\beta \text{ nt } \}$$

- $0 < \beta \leq \beta_E$: d^β nt.
- $\beta > \beta_E$: d^β non nt.

Some values of β_E

- $\beta_E < \infty$ except particular case
- Metric square (side=1, diagonal $0 < d \leq 2$)
  - $0 < d \leq 1$: $\beta_E = +\infty$
  - $1 < d \leq 2$: $\beta_E = 1/\log_2 d$
- Euclidean space $(\mathbb{R}^n, ||.||_{\ell^2})$: $\beta_E = 2$
- Smooth manifold: $\beta_E \leq 2$
- Smooth manifold, curvature > 0: $\beta_E < 2$

Some values of β_E (ctd)

- Sphere + geodesic distance: $\beta_E = 1$
- Hyperbolic space + geodesic distance: $\beta_E = 1$
- Tree + geodesic distance: $\beta_E \geq 1$
- $(\mathbb{R}^n, ||.||_{\ell^q})$ with $n \geq 3$, $q > 2$: $\beta_E = 0$

Distances defined by a measure (dm)

$(M, N) \mapsto d(M, N)$ is dm if

there exists a measure μ

and an application $M \mapsto H_M, \ E \to \mathbf{H}$ such that

$$d(M, N) \quad = \quad \mu(H_M \Delta H_N) \ ,$$

Δ: symmetric difference between sets

- dm $\Rightarrow$ nt

Examples of distances dm

- Euclidean norm $\| \|^{\beta}$ with $0 < \beta \leq 1$ (Chentsov, Takenaka)

- Geodesic distance on spheres (Lévy)

- Geodesic distance on hyperbolic spaces (Takenaka & Kubo & Urakawa, Robertson)

- Geodesic distance on trees (Valette)

Gaussian fields

An other consequence of Schönberg's theorem

d^β nt $\Leftrightarrow (d^\beta(O,M) + d^\beta(O,N) - d^\beta(M,N))$ nnd

FBM exists iff $0 < H \leq \beta_E/2$

The Euclidean case $\beta_E = 2$ is atypical.

Stable fields (spheres, hyperbolic spaces)

Existance iff $0 < H \leq 1/\alpha$

- $H > 1/\alpha$: no (Schönberg)
- $1/2 \leq H \leq 1/\alpha$: dm + Chentsov-Takenaka's construction
- $0 < H \leq 1/2$: new construction

Multifractional brownian motion

$|t|^{H(t)+H(s)} + |s|^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}$ nnd?

$c_{H(t)+H(s)}\big(|t|^{H(t)+H(s)} + |s|^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\big)$

$$c_H = \int_{\mathbb{R}} \frac{|e^{iy}-1|^2}{|y|^{2H}} \frac{dy}{|y|}$$

always nnd.

Multifractional brownian fields (ctd)

$$\mu_n(dx) \quad = \quad \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{dx_1 \ldots dx_n}{\|x\|^n},$$

unique, up to a constant, measure invariant by homotheties and rotation

μ, extension of μ_n to $\mathbb{R}^\infty$

invariant by homotheties and ℓ^2-rotation

$$C_H = \int_{\ell^2} \frac{|e^{i<(1,0,0,\ldots),y>} - 1|^2}{\|y\|^{2H}} \mu(dy)$$

Multifractional brownian fields: covariance

d of negative type, $H(M) \in (0, 1/2)$:

$$\begin{aligned} R(M,N) &= C_{H(M)+H(N)}(d^{H(M)+H(N)}(O,M) \\ &\quad + d^{H(M)+H(N)}(O,N) - d^{H(M)+H(N)}(M,N)), \end{aligned}$$