The energy density in the 2D Ising model.

Clément Hongler.
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Joint work with Stas Smirnov.

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The Ising model on a graph G

- Probability space: $\{\pm 1\}^{V(G)}$
- For $\vec{\sigma} \in \{\pm 1\}^{V(G)}$,

  $\mathbb{P}[\vec{\sigma}] = \frac{1}{Z_\beta} e^{-\beta H(\vec{\sigma})}$

- $\beta > 0$ (inverse temperature)
- $H(\vec{\sigma}) = -\sum_{\langle x,y\rangle \in E(G)} \sigma_x \sigma_y$ (energy)

- $Z_\beta = \sum_{\vec{\sigma} \in \{\pm 1\}^{V(G)}} e^{-\beta H(\vec{\sigma})}$

  (the partition function)

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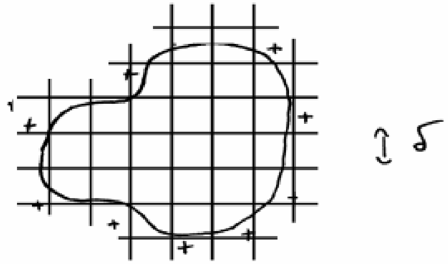
The Ising model on the square grid.

- Ω a Jordan domain.



- $\Omega^\delta := \Omega \cap \delta \mathbb{Z}^2$

(largest connected component)

- Ising model on Ω^δ with + boundary condition
- What happens as $\delta \to 0$?

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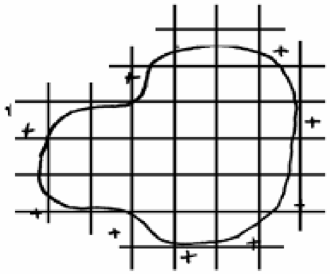

$\updownarrow \delta$

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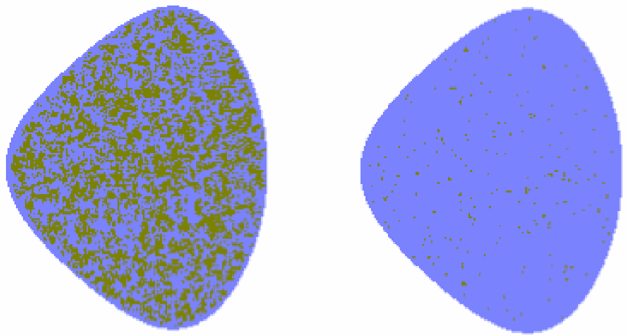
- What happens as $\delta \to 0$?

It depends on β !
($\mathbb{P}[\vec{\sigma}] \propto e^{-\beta H(\vec{\sigma})}$)

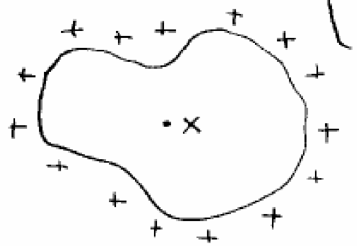


$\beta < \beta_c = \frac{1}{2}\ln(\sqrt{2}+1)$

$\beta > \beta_c$

Thm (Onsager, 1944)

$$\mathbb{E}[\sigma_x] \xrightarrow{\delta \to 0} \begin{cases} 0 & \text{if } \beta \le \beta_c \\ c(\beta) > 0 & \text{if } \beta > \beta_c \end{cases}$$



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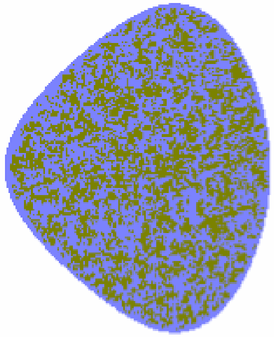

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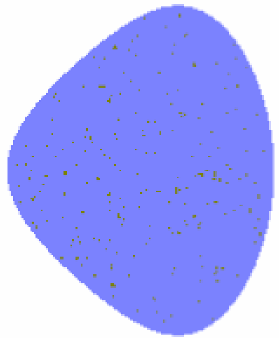

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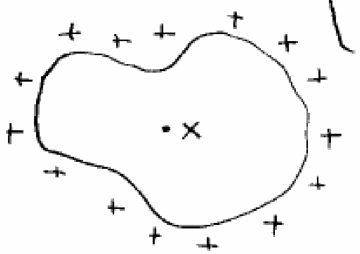
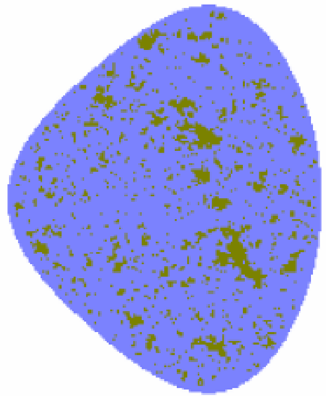


What happens at β_c ?



- Universality
- Conformal invariance

Conformal Field Theory → Understand local observables

Schramm Loewner Evolution. → Understand global geometry

Effect of the shape of Ω .

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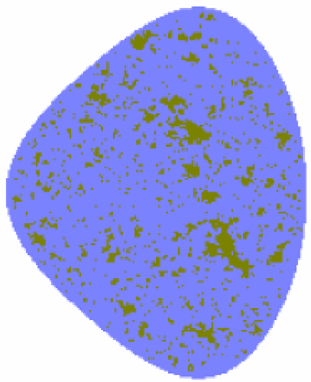

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$\downarrow$

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The energy density.

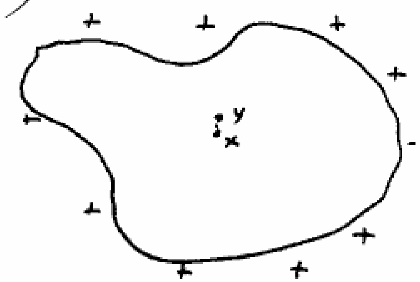
The energy terms in

$$H(\vec{\sigma}) = -\sum_{\langle x,y\rangle \in E(G)} \sigma_x \sigma_y .$$

Question: how does the energy distributes accross the lattice? Fix an edge $a = \langle x,y \rangle$.

Thm (H., Smirnov)

$$\frac{\mathbb{P}[\sigma_x = \sigma_y] - \frac{2+\sqrt{2}}{4}}{\delta} \xrightarrow{\delta \to 0} \frac{1}{4\pi} \ell_\Omega(a)$$



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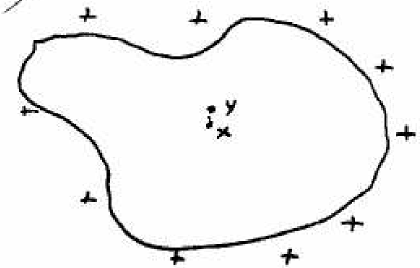

The hyperbolic metric element

$$\ell_{\mathbb{D}(0,1)}(a) = \frac{1}{1-|a|^2}.$$

(M.C. Escher)

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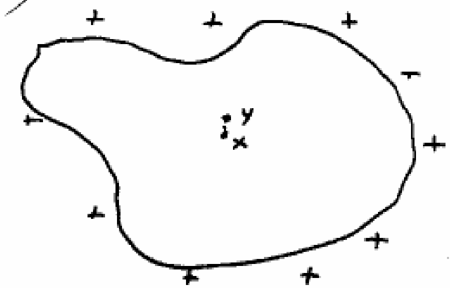

Sketch of the proof.

(A) Contour representation

(B) Holomorphic deformation.

(C) Limit $\delta \to 0$.

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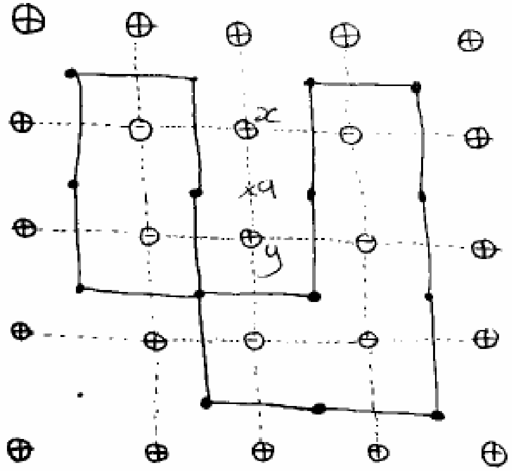

Bijection $\{\pm 1\}^{V(G)} \longleftrightarrow \mathcal{C}$

$\mathcal{C} := \{\gamma \subseteq E(G^*) : \gamma \text{ coll of closed loops}\}$

$\mathbb{P}[\gamma] = \frac{1}{Z} \alpha^{\#\text{edges}(\gamma)}$

$\alpha = \alpha_c = \sqrt{2} - 1.$

$Z = \sum_{\gamma \in \mathcal{C}} \alpha^{\#\text{edges}(\gamma)}$

Identify a to its middlepoint

$\mathcal{C}_a = \{\gamma \in \mathcal{C} : a \notin \gamma\}$

$Z_a = \sum_{\gamma \in \mathcal{C}_a} \alpha^{\#\text{edges}(\gamma)}$

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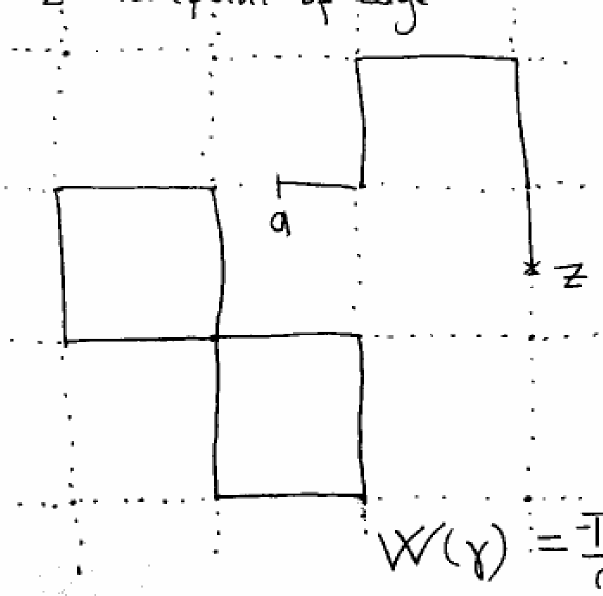
Idea: construct a (discrete) holomorphic deformation of $\frac{Z_a}{Z}$

$C_{a,z} = \{\gamma \subseteq E(G) : \gamma$ contains closed loops and a path from a to z starting from left to right at a$\}$

z : middlepoint of edge



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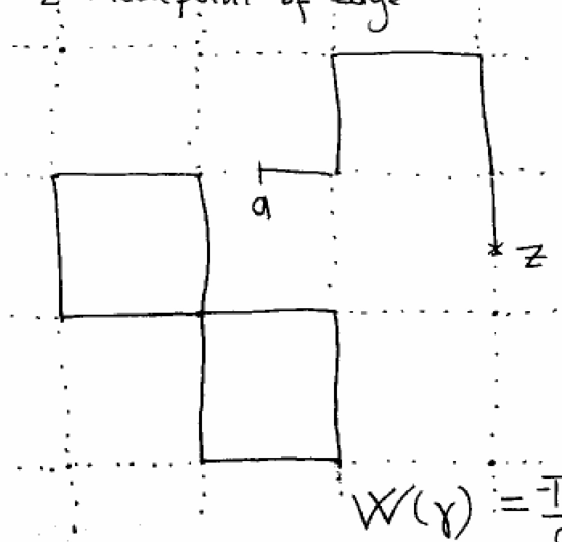

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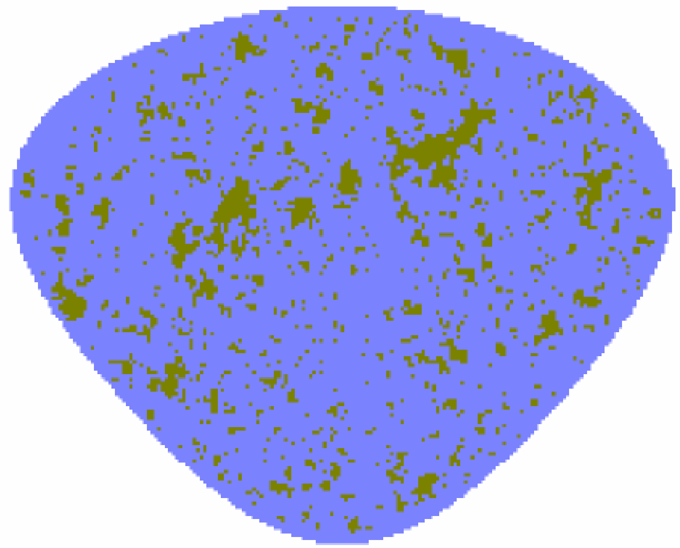

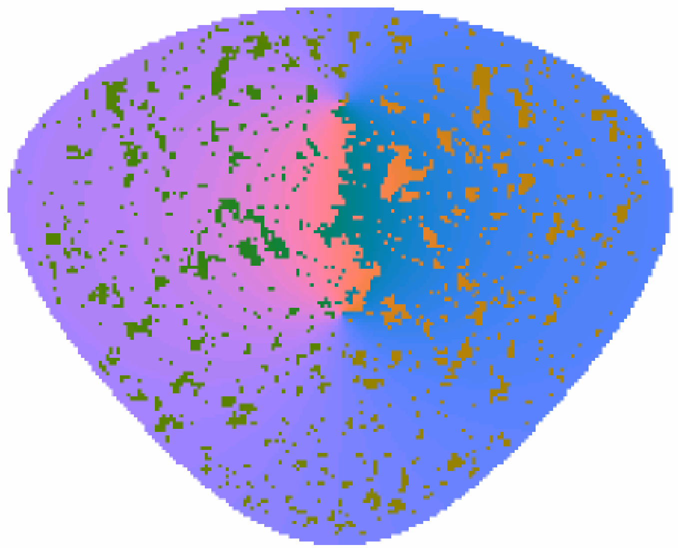

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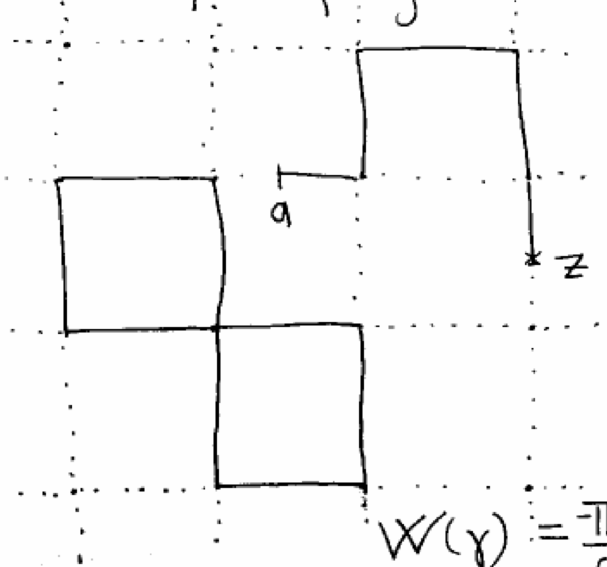

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W : winding number of the path from a to z.
(total rotation of the path).

Holomorphic observable.

$$f_a^\delta(z) := \frac{\sum_{\gamma \in C_{a,z}^\delta} \alpha^{\#\text{edges}(\gamma)} e^{-i\frac{W(\gamma)}{2}}}{Z^\delta}$$

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Properties of $z \longmapsto f_a^\delta(z)$

(1) Discrete holomorphic on $\Omega \setminus \{a\}$
$\Longrightarrow \bar\partial^\delta f_a^\delta = 0$, $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$

(2) Discrete singularity at a.
$(\bar\partial^\delta f_a^\delta)(a) = \frac{1}{2}$

(3) Boundary condition:
$f_a^\delta|_{\partial\Omega} \parallel \frac{1}{\sqrt{n}}$; n: normal to ext.

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Derivation of the energy density

$(1) + (2) \Longleftrightarrow h_a^\delta := f_a^\delta - g_a^\delta$

discrete analytic on Ω;

g_a^δ discrete Green function

$g_a^\delta(a) = \frac{\sqrt{2}+2}{4}$.

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Proof of convergence

1) Precompactness.

(difficult: control a function knowing only its argument)

2) Identification of limit: integral trick

$f_a^\delta \parallel \frac{1}{\sqrt{n}}$ $\quad (f_a^\delta)^2 \parallel \frac{1}{n}$

$$\operatorname{Re}\left(\int (f_a^\delta)^2\right)\Big|_{\partial\Omega} = cst$$

Dirichlet condition: better

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Proof of precompactness

- Assume $\partial\Omega$ of finite length

Idea: control $H_a^\delta := \mathrm{Re}\left(\int (h_a^\delta)^2\right)$

- H_a^δ subharmonic

$\partial_n H_a^\delta \leq M$ on $\partial\Omega$.

$0 \leq \sum\sum \Delta H_a^\delta = \sum (\partial_n H_a^\delta)_+ - (\partial_n H_a^\delta)_-$

$\sum (\partial_n H_a^\delta)_- \leq \sum (\partial_n H_a^\delta)_+$

$\leq M\, \ell(\partial\Omega)$

$\sum |\partial_n H_a^\delta| \leq 2M\ell(\partial\Omega)$

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$H_a^\delta = H + S$, with

$\Delta H = 0$, $S|_{\partial\Omega} = 0$.

$H|_{\partial\Omega} = H_a^\delta|_{\partial\Omega}$ $\Delta S \ge 0$.

—

Control of S follows from $\sum\sum \Delta S \le M\ell(\partial\Omega)$

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Control of H follows from $\sum |\partial_n H| \le 2M\ell(\partial\Omega)$

Idea: pass to harmonic conjugate $\rightsquigarrow \sum |\partial_t C|$ bd

$\Rightarrow C$ bd.

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Thank you!