# Asymptotics of solutions to the fragmentation equation: an approach via self-similar Markov processes

(to appear in the Annals of Applied Probability)

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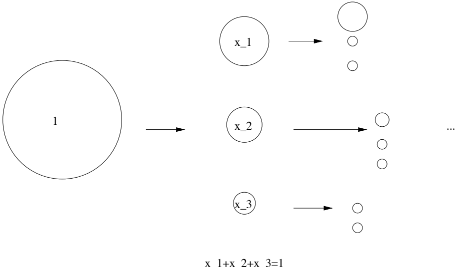
Self-similar processes and their applications, Angers 2009

# Outline

1. The fragmentation equation
2. Construction of solutions via time-changed subordinators
3. Large times behavior

# The fragmentation equation

- models systems of particles underlying fragmentation
- conservation of mass: when a mass $x$ splits in masses $x_1, x_2, \ldots : \sum_{i=1}^{\infty} x_i = x$




$$\partial_t n_t(x) = \int_x^{\infty} a(y) b(y,x) n_t(y) \mathrm{d}y - a(x) n_t(x),$$

• $n_t(x)$: concentration of particles of mass $x$ at time $t$
• $a(x)$: break-up rate of a particle with mass $x$
• $b(x,y)$: distribution of masses $y$ produced by the fragmentation of a particle of mass $x$ ($y < x$)

Conservation of mass: $\int_0^x y b(x,y) \mathrm{d}y = x$.

# The fragmentation equation

Self-similar dynamic:

- $a(x) = Cx^{\alpha}$, for some $C > 0$ and $\alpha \in \mathbb{R}$
- $b(x, y) = h(y/x)/x$, with $\int_0^1 yh(y)\mathrm{d}y = 1$

$$\Rightarrow \partial_t n_t(x) = \int_x^{\infty} y^{\alpha-1} Ch(x/y) n_t(y)\mathrm{d}y - Cx^{\alpha} n_t(x),$$

Weak form:

$$\partial_t < \mu_t, f >= \int_0^{\infty} x^{\alpha} \left( \int_0^1 (f(yx) - f(x)y)\, Ch(y)\mathrm{d}y \right) \mu_t(\mathrm{d}x),$$

where $\mu_t(\mathrm{d}x) = n_t(x)\mathrm{d}x$ and $f$ is a test function.

# Generalized weak form

For any test function $f$,

$$\partial_t < \mu_t, f >= \int_0^\infty x^\alpha \left( \int_0^1 (f(yx) - f(x)y) \, B(\mathrm{d}y) \right) \mu_t(\mathrm{d}x), \quad \text{(Eq. Frag.)}$$

- $(\mu_t, t \geq 0)$: measures on $(0, \infty)$ $\quad \alpha \in \mathbb{R}$
- $B$ measure on $(0, 1)$ such that

$$\int_0^1 y(1-y)B(\mathrm{d}y) < \infty, \text{ and } B((0,1)) > 0.$$

Note: $\int_0^1 xB(\mathrm{d}x)$ may be infinite !

Standard weak form $\Leftrightarrow$ $B(\mathrm{d}y) = Ch(y)\mathrm{d}y$ with $\int_0^1 yh(y)\mathrm{d}y = 1$

## Solutions to (Eq. frag.)

- avoid non-physical solutions

### Definition

Let $\mu_0 : \int_0^\infty x\mu_0(\mathrm{d}x) = 1$. A family $(\mu_t, t \geq 0)$ is *a solution to the fragmentation equation* if

1. $(\mu_t, t \geq 0)$ satisfies (Eq. Frag) for all test function $f$ $C^1$ with compact support in $(0, \infty)$
2. $m(t) := \int_0^\infty x\mu_t(\mathrm{d}x) \leq m(0) = 1$ for all $t \geq 0$
3. $\mu_0([M, \infty)) = 0 \Rightarrow \mu_t([M, \infty)) = 0$ for all $t \geq 0$

Note: self-similarity of solutions: if $(\mu_t, t \geq 0)$ is a solution to (Eq. Frag), so is $(\gamma^{-1}\mu_{t\gamma^\alpha} \circ (\gamma \mathrm{id})^{-1})$, $\forall \gamma > 0$.

# Construction of solutions via subordinators

- $\mu_0$ measure on $(0,\infty)$: $\int_0^\infty x\mu_0(\mathrm{d}x) = 1$, $\quad X(0)$ r.v. $\sim x\mu_0(\mathrm{d}x)$

- $\Pi$ measure on $(0,\infty)$: $\int_0^\infty g(x)\Pi(\mathrm{d}x) = \int_0^1 g(-\ln(x))xB(\mathrm{d}x)$

$\xi$ subordinator with zero drift, Lévy measure $\Pi$, independent of $X(0)$

- $X(t) := X(0)\exp(-\xi_{\rho(X(0)^\alpha t)})$ with $\rho(t) := \inf\left\{u : \int_0^u \exp(\alpha\xi_r)\mathrm{d}r > t\right\}$

- $\int_{(0,\infty)} f(x)x\mu_t(\mathrm{d}x) := \mathbb{E}[f(X(t))]$, $\forall\, f : [0,\infty[\to [0,\infty[$, $f(0) = 0$

$$X(t) \sim x\mu_t(\mathrm{d}x) + (1 - m(t))\delta_0(\mathrm{d}x)$$

**Theorem**

*(i)* $(\mu_t, t \geq 0)$ *is a solution to (Eq. Frag) as soon as* $\alpha \leq 0$ *or* $\alpha > 0$ *and* $\int_1^\infty x\ln(x)\mu_0(\mathrm{d}x) < \infty$.

*(ii) This solution is unique as soon as* $\mu_0([M,\infty[) = 0$ *for some* $M > 0$.

# Dust

Total mass of non-zero particles:

$$m(t) = \int_0^\infty x\mu_t(\mathrm{d}x) = \mathbb{P}(X(t) > 0)$$

1. $\alpha \geq 0$ : $\rho(t) \leq t \Rightarrow X(t) > 0$ a.s. and $m(t) = 1\ \forall t \geq 0$
2. $\alpha < 0$ : $\rho(t) = \infty$ for $t \geq \int_0^\infty \exp(\alpha \xi_r)\mathrm{d}r := I$

   $\Rightarrow X(t) = 0 \quad \Leftrightarrow \quad X(0)^\alpha t \geq I$

   $\Rightarrow m(t) = \mathbb{P}(I > X(0)^\alpha t) \to 0$ as $t \to \infty$.

$I$ : exponential functional of a subordinator (Carmona-Petit-Yor 97, Bertoin-Yor 01, Rivero 03)

## Our goal is to describe:

- the behavior of $\mu_t$ as $t \to \infty$

  or equivalently

- the behavior as $t \to \infty$ of $m(t)$ and $X(t) = X(0)\exp(-\xi_{\rho(X(0)^\alpha t)})$

conditioned on non-extinction $\left( X(t)|X(t) > 0 \overset{\text{law}}{\sim} x\mu_t(\mathrm{d}x)/m(t) \right)$

For $\alpha > 0$: if $\int_0 |\ln(x)| x B(\mathrm{d}x) < \infty$ and $\mu_0 = \delta_1$ ($X(0) = 1$ a.s.)

$$t^{1/\alpha} X(t) \xrightarrow{\mathrm{d}} N(\infty)$$

- analytic approach: ESCOBEDO-MISCHLER-RODRIGUEZ RICARD (05)
- probabilistic approach: BERTOIN-CABALLERO (02)

## Large time behavior of $m$ when $\mu_0 = \delta_1$, i.e. $X(t) = \exp(-\xi_{\rho(t)})$

### Hypothesis (H)

$\exists \beta \in [0,1)$ and $0 < C < \infty$ such that

$$\int_0^{1-u} xB(\mathrm{d}x) \underset{u \to 0}{\sim} Cu^{-\beta} \quad \left(\Leftrightarrow \int_u^\infty \Pi(\mathrm{d}x) \underset{u \to 0}{\sim} Cu^{-\beta}\right)$$

(possible extension to regular variation)

### Proposition

*Assume* $\alpha < 0$. *Then,*

1. $0 < m(t) < 1$ *for all* $t > 0$
2. $m(t) \leq C_\lambda \exp(-\lambda t) \quad \forall t \geq 0, \forall \lambda < \int_0^1 xB(\mathrm{d}x)$
3. *if (H) holds,*

$$-\ln(m(t)) \underset{t \to \infty}{\sim} (1-\beta) C' t^{1/(1-\beta)}$$

   *where* $C' = \left(C|\alpha|^\beta \Gamma(1-\beta)\right)^{1/(1-\beta)}$
4. $m \in \mathcal{C}^\infty([0,\infty))$ *as soon as* $\int_0 x|\ln(x)|\mathrm{B}(dx) < \infty$ $(\Leftrightarrow \mathbb{E}[\xi_1] < \infty)$

## Large time behavior of $(\mu_t, t \geq 0)$ when $\mu_0 = \delta_1$

### Theorem

*Suppose (H), $\alpha < 0$ and $\int_0 |\ln(x)| x B(\mathrm{d}x) < \infty$. Then for all continuous bounded test functions $f : (0,\infty) \to \mathbb{R}$,*

$$\frac{1}{m(t)} \int_0^\infty f\left(\left(C' t^{\beta/(1-\beta)}\right)^{1/|\alpha|} x\right) x\mu_t(\mathrm{d}x) \underset{t\to\infty}{\to} \int_0^\infty f(x) x \mu_\infty(\mathrm{d}x),$$

*where $x\mu_\infty(\mathrm{d}x)$ is a probability on $(0,\infty)$ characterized by its moments*

$$\int_0^\infty x^{|\alpha| n} x\mu_\infty(\mathrm{d}x) = \phi(|\alpha|)\phi(2|\alpha|)...\phi(n|\alpha|), \; n \geq 1,$$

*where*

$$\phi(t) = \int_0^1 (1 - x^t) x B(\mathrm{d}x).$$

**Remark.** (BERTOIN-YOR 01) $\exists$ a r.v. $R$ independent of $I$ such that

$$RI \overset{d}{=} \mathbf{e}(1).$$

Here: $x\mu_\infty(\mathrm{d}x)$: distribution of $R^{1/|\alpha|}$.

Interpretation: typical particle conditioned on non-extinction: $M(t) \overset{d}{=} X(t) | X(t) > 0$, then

$$\left(C' t^{\beta/(1-\beta)}\right)^{1/|\alpha|} M(t) \overset{d}{\to} M(\infty) \sim x \mu_\infty(\mathrm{d}x)$$

Comparison with cases where $\alpha > 0$: when $\int_0 |\ln(x)| x B(\mathrm{d}x) < \infty$,

$$t^{1/\alpha} M(t) \overset{d}{\to} N(\infty).$$

## Sketch of proof:

- when $I = \int_0^\infty \exp(\alpha\xi_r)\mathrm{d}r > t$,

$$I = \int_0^{\rho(t)} \exp(\alpha\xi_r)\mathrm{d}r + \exp(\alpha\xi_{\rho(t)}) \int_0^\infty \exp(\alpha(\xi_{r+\rho(t)} - \xi_{\rho(t)}))\mathrm{d}r$$

hence: $(I-t)^+ = \exp(\alpha\xi_{\rho(t)})\tilde{I}$, where $\tilde{I} \overset{\text{law}}{=} I$ and is independent of $\exp(-\xi_{\rho(t)})$.

- for all $n \in \mathbb{N}^*$,

$$\begin{aligned} & \mathbb{P}(X(t)>0)^{-1}\mathbb{E}\left[\left(\left(\frac{-\ln m(t)}{(1-\beta)t}\right)^{1/|\alpha|} X(t)^{|\alpha|n}\right)\right]\mathbb{E}[I^n] \\ = \ & m(t)^{-1}\mathbb{E}\left[\left(\left(\frac{-\ln m(t)}{(1-\beta)t}\right)^{1/|\alpha|} \exp(-\xi_{\rho(t)})\right)^{|\alpha|n}\right]\mathbb{E}[I^n] \\ = \ & m(t)^{-1}\left(\frac{-\ln m(t)}{(1-\beta)t}\right)^{n}\mathbb{E}\left[((I-t)^+)^n\right] \\ = \ & nm(t)^{-1}\left(\frac{-\ln m(t)}{(1-\beta)t}\right)^{n}\int_t^\infty (x-t)^{n-1}m(x)\mathrm{d}x \end{aligned}$$

and this $\to n!$ as $t \to \infty$, using (H)

• Since $\mathbb{E}[R^n]\mathbb{E}[I^n] = n!$, $(RI \overset{\text{law}}{=} \mathbf{e})$

$$\mathbb{E}\left[\left(\left(\frac{-\ln m(t)}{(1-\beta)t}\right)^{1/|\alpha|} X(t)\right)^{|\alpha| n} \middle| X(t) > 0\right] \underset{t\to\infty}{\to} \frac{n!}{\mathbb{E}[I^n]} = \mathbb{E}[R^n].$$

• Conclusion: convergence in distribution since the law of $R$ is characterized by its moments. □

## Quasi-stationary solutions

### Definition

A quasi-stationary solution is a solution $(\mu_t, t \geq 0)$ such that

$$\mu_t = m(t)\mu_0, \quad \forall t \geq 0, \quad (\text{ with } m(t)=\int_0^\infty x\mu_t(\mathrm{d}x)).$$

Consider $\mu_\infty$ and its self-similar counterparts:

$$\mu_\infty^{(\lambda)} := \lambda^{-1}\mu_\infty \circ (\lambda \mathrm{id})^{-1}, \quad \lambda > 0.$$

### Theorem

*For all $\lambda > 0$, let $(\mu_{\infty,t}^{(\lambda)}, t \geq 0)$ denote the solution to the fragmentation equation starting from $\mu_\infty^{(\lambda)}$ and constructed via a subordinator. Then*

$$\mu_{\infty,t}^{(\lambda)} = \exp(-\lambda^\alpha t)\mu_\infty^{(\lambda)} = m(t)\mu_\infty^{(\lambda)}, \quad \forall t \geq 0.$$

*Reciprocally, if $(\mu_t, t \geq 0)$ is a quasi-stationary solution, then $\exists \lambda > 0$:*

$$(\mu_t, t \geq 0) = (\mu_{\infty,t}^{(\lambda)}, t \geq 0)$$