The exit problem for certain Lévy processes

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If X is a subordinator then $A \supset \{\lambda; \Re(\lambda) \geq 0\}$
If $-X$ is a subordinator then $A \supset \{\lambda; \Re(\lambda) \leq 0\}$.

Theorem

There exist an exponent of (possibly killed) subordinator ψ and of the opposite of a subordinator $\hat{\psi}$ such that

$$\hat{\psi}(\lambda).\psi(\lambda) = \phi(\lambda) \qquad \text{for} \quad \lambda \in i\mathbf{R}$$

Such a couple is unique up to a multiplicative constant.

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If $\psi(0) > 0$ (ie. X dies or derives to $-\infty$) , then for $M := \sup\{X_s, s \in [0, \zeta[\}$*;*

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$$\psi(\lambda) \int_{]x,+\infty[} e^{-\lambda y} U(dy) =: \psi(\lambda) U_{]x,+\infty[}(\lambda) = \mathbf{E}(e^{-\lambda X_{T^x}} 1_{T^x < \zeta})$$

An explicit case of W-H factorization

If $$1_{y>0}\pi(dy) = \sum_{\text{finite}} c_j y^{n_j} e^{-\gamma_j y} 1_{y>0} dy, \qquad c_j \in \mathbf{C}, \Re(\gamma_j) > 0.$$

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$\frac{\psi(\lambda)}{\lambda}$ and $\frac{1}{\psi(\lambda)}$ bounded $\Longrightarrow \Psi$ = Polynomial of degree n or $n+1$

$$\Longrightarrow \psi(\lambda) = \psi_\infty \cdot \frac{\prod_{i=1}^{m}(\lambda + \beta_i)}{\prod_{i=1}^{n}(\lambda + \gamma_j)}$$

And the zeros of ψ, $-\beta_1, \dots, -\beta_m$, are the zeros of ϕ belonging to $\{\Re(\lambda) < 0\}$.

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$$\Longrightarrow \mathbf{P}(M \in dy) = \psi(0) U(dy) = a_0 \delta_0(dy) + \sum a_i y^{m_i} e^{-\beta_i y} 1_{y>0} dy$$

$$a_0 + \sum \frac{a_i}{(\lambda + \beta_i)^{m_i+1}} = \frac{\psi(0)}{\psi(\lambda)}$$

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$$e^{-\lambda x}(c_0(x) + \sum \frac{c_j(x)}{(\lambda + \gamma_j)^{n_j+1}}) = \psi(\lambda) U_{[x,+\infty[}(\lambda)$$

$$= \frac{\psi(\lambda)}{\psi(0)} . \int_{]x,+\infty[} \sum a_i y^{m_i} e^{-(\beta_i + \lambda) y} dy$$

A matrix W-H Factorization and the exit problem

For $x \in]0, +\infty]$ there exist $A(x,\lambda)$, $\hat{A}(x,\lambda)$, $C(x,\lambda)$, $\hat{C}(x,\lambda)$, Laplace transforms of measure supported by $[0,x]$, $[-x,0]$, $[x,+\infty[$, and $]-\infty,-x]$,

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Let $m := \inf\{X_s; s \in [0,\zeta[\}$, $M := \sup\{X_s; s \in [0,\zeta[\}$, $F := X_\zeta$,

$$\mathbf{E}(e^{-\mu m}; M - m \leq x, e^{-\lambda(F-m)}) = \phi(0).\hat{A}(x,\mu).A(x,\lambda)$$

$+$ Distributions of (I_{V^x}, X_{V^x}), (S_{V_x}, X_{V_x}), (I_{U^x}, S_{U^x}), $(I_{T_a^b}, S_{T_a^b})$ $(S_t = \sup\{X_s; s \in [0,t]\} \qquad I_t = \inf\{X_s; s \in [0,t]\})$.

$$V^x := \inf\{t; X_t - I_t > x\}, \qquad V_x := \inf\{t; X_t - S_t < -x\}$$

$$U^x := \inf\{t; S_t - I_t > x\} \qquad T_a^b = \inf\{t; X_t \notin [-a, b]\}$$

are expressed by the functions $A, B, C, \hat{A}, \hat{B}, \hat{C}$.

First example : $\phi = \hat{\psi}$

$$\begin{pmatrix} \hat{U}_{[-x,0]}(\lambda) & -e^{\lambda x} \\ e^{-\lambda x}\hat{\psi}(\lambda)\hat{U}_{]-\infty,-x[}(\lambda) & \hat{\psi}(\lambda) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ e^{-\lambda x}\hat{U}_{[-x,0]}(\lambda) & 1 \end{pmatrix}$$

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For $\phi(\lambda) = \hat{\psi}(\lambda)\frac{\Pi_{i=1}^{m}(\lambda+\beta_i)}{\Pi_{j=1}^{n}(\lambda+\gamma_j)}$, notice :

$$\begin{pmatrix} \hat{A}(x,\lambda) & -e^{\lambda x}A(x,\lambda) \\ e^{-\lambda x}\hat{C}(x,\lambda) & \hat{B}(x,\lambda) \end{pmatrix} \cdot \begin{pmatrix} A(x,\lambda) & -e^{\lambda x}C(x,\lambda) \\ e^{-\lambda x}\hat{A}(x,\lambda) & B(x,\lambda) \end{pmatrix}$$

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$$= \begin{pmatrix} \Pi(\lambda+\gamma_j) & 0 \\ 0 & \Pi(\lambda+\beta_i) \end{pmatrix} \begin{pmatrix} 0 & -e^{-\lambda x} \\ e^{\lambda x} & \hat{\psi}(\lambda) \end{pmatrix} \begin{pmatrix} \frac{1}{\Pi(\lambda+\beta_i)} & 0 \\ 0 & \frac{1}{\Pi(\lambda+\gamma_j)} \end{pmatrix}$$

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$P_1 P_2 - Q_1 Q_2 = \Pi(\lambda+\gamma_j)\Pi(\lambda+\beta_i)$

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$\Longrightarrow$ computation of $P_1(x,\lambda)$ and $Q_1(x,\lambda)$.

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$A(x,\lambda)$ and $\hat{B}(x,\lambda)$ are holomorphic on $\{Re(\lambda) < 0\}$ and $A(x,\lambda) \sim \frac{1}{\psi(\lambda)}$ for $\lambda \to \infty$ then

$$P_1(x,-\beta_i) + Q_1(x,\beta_i)e^{\beta_i x}\hat{U}_{[-x,0]}(-\beta_i) = 0$$

$$P_1(x,-\gamma_j) + Q_1(x,\gamma_j)e^{\gamma_j x}\hat{U}_{]-\imath,-x[}(-\gamma_j) = 0$$

$$P_1(x,\lambda) \sim \frac{1}{\psi_\infty}.\lambda^n$$

$\Longrightarrow$ computation of $P_1(x,\lambda)$ and $Q_1(x,\lambda)$.

$$A(x,\lambda) = a_0 + \sum a_i(x)\frac{e^{-\lambda x}\hat{U}1_{[-x,0]}(\lambda) - e^{\beta_i x}\hat{U}1_{[-x,0]}(-\beta_i)}{\lambda+\beta_i}$$

(if the β_i are all different)

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

$+$ expressions of $\mathbf{P}(M - m \leq x)$ that involves β_i, γ_j, $e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

$+$ expressions of $\mathbf{P}(M - m \leq x)$ that involves β_i, γ_j,
$e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.
$+$ The distributions of
$X_{V^x} - I_{V^x} - x$,
$X_{V^x} - x$ given $\{X_{V^x} > x\}$,($V^x = \inf\{t, X_t - I_t > x\}$)

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

+ expressions of $\mathbf{P}(M - m \leq x)$ that involves β_i, γ_j,
$e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.
+ The distributions of
$X_{V^x} - I_{V^x} - x$,
$X_{V^x} - x$ given $\{X_{V^x} > x\}$,($V^x = \inf\{t, X_t - I_t > x\}$)
$X_{U^x} - I_{U^x} - x$, given $X_{U^x} = S_{U^x}$,
$X_{U^x} - x$ given $X_{U^x} - x > 0$ $\qquad (U^x = \inf\{t, S_t - I_t > x\})$.

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

+ expressions of $\mathbf{P}(M - m \leq x)$ that involves β_i, γ_j, $e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.

+ The distributions of

$X_{V^x} - I_{V^x} - x$,

$X_{V^x} - x$ given $\{X_{V^x} > x\}$,($V^x = \inf\{t, X_t - I_t > x\}$)

$X_{U^x} - I_{U^x} - x$, given $X_{U^x} = S_{U^x}$,

$X_{U^x} - x$ given $X_{U^x} - x > 0$ $\quad (U^x = \inf\{t, S_t - I_t > x\})$.

$X_{T_a^b} - b$ given $X_{T_a^b} - b > 0$ $\quad (T_a^b = \inf\{t, X_t \notin [-a, b]\}$.

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \leq x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy - x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

+ expressions of $\mathbf{P}(M - m \leq x)$ that involves β_i, γ_j,
$e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.
+ The distributions of
$X_{V^x} - I_{V^x} - x$,
$X_{V^x} - x$ given $\{X_{V^x} > x\}$,($V^x = \inf\{t, X_t - I_t > x\}$)
$X_{U^x} - I_{U^x} - x$, given $X_{U^x} = S_{U^x}$,
$X_{U^x} - x$ given $X_{U^x} - x > 0$ $\qquad (U^x = \inf\{t, S_t - I_t > x\})$.
$X_{T_a^b} - b$ given $X_{T_a^b} - b > 0$ $\qquad (T_a^b = \inf\{t, X_t \notin [-a, b]\}$.
There are all combinations of measures of the form
$y^{n_j} e^{-\gamma_j y} 1_{y>0} dy$ with explicit coefficients.

$$\Longrightarrow \mathbf{P}(M \in dy | M - m \le x)$$

$$= a_0(x)\delta_0(dy) + 1_{y\in[0,x]} \sum a_i(x)[1_{y\in[0,x]} U(dy-x)] * [y^{m_i} e^{-\beta_i y} 1_{y\in[0,x]}]$$

$+$ expressions of $\mathbf{P}(M - m \le x)$ that involves β_i, γ_j, $e^{\beta_i x} \check{U} 1_{[-x,0]}(-\beta_i)$ and $e^{\gamma_j x} \check{U} 1_{]-\infty,-x}(-\gamma_j))$.

$+$ The distributions of

$X_{V^x} - I_{V^x} - x$,

$X_{V^x} - x$ given $\{X_{V^x} > x\}$,($V^x = \inf\{t, X_t - I_t > x\}$)

$X_{U^x} - I_{U^x} - x$, given $X_{U^x} = S_{U^x}$,

$X_{U^x} - x$ given $X_{U^x} - x > 0$ $\quad$ $(U^x = \inf\{t, S_t - I_t > x\})$.

$X_{T_a^b} - b$ given $X_{T_a^b} - b > 0$ $\quad$ $(T_a^b = \inf\{t, X_t \notin [-a, b]\}$.

There are all combinations of measures of the form $y^{n_j} e^{-\gamma_j y} 1_{y>0} dy$ with explicit coefficients.

$+$ The distribution of S_{V_x}, $V_x = \inf\{t; X_t - S_t < -x\}$), of the form $c_0(x)\delta_0(x) + \sum c_i(x).y^{m_i(x)} e^{-\beta_i(x) y} 1_{y>0}$ (with explicit parameters $\beta_i(x)$ and coefficients $c_i(x)$),