The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Valuation of exotic options in Lévy models

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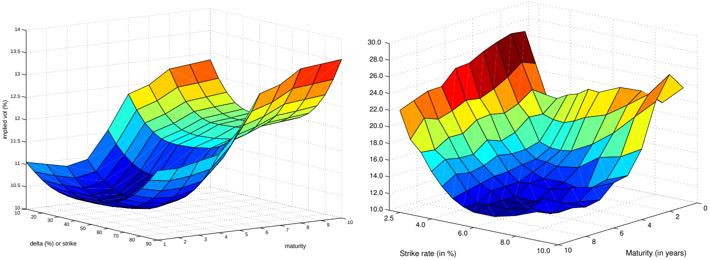
Volatility smile and surface



Volatility surfaces of foreign exchange and interest rate options

- Volatilities vary in strike (*smile*)
- Volatilities vary in time to maturity (*term structure*)
- Volatility clustering

Exponential semimartingale model

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

$\mathcal{B}_T = (\Omega, \mathcal{F}, \mathbf{F}, P)$ stochastic basis, where $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T}$.
Price process of a financial asset as exponential semimartingale

$$S_t = S_0 \mathrm{e}^{H_t}, \quad 0 \le t \le T. \tag{1}$$

$H = (H_t)_{0 \le t \le T}$ semimartingale with canonical representation

$$H = B + H^c + h(x) * (\mu^H - \nu) + (x - h(x)) * \mu^H. \tag{2}$$

For the processes B, $C = \langle H^c \rangle$, and the measure ν we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which is called the *triplet of predictable characteristics* of H.

Alternative model description

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

$\mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \leq t \leq T}$ stochastic exponential

$$\begin{aligned} S_t &= \mathcal{E}(\widetilde{H})_t, \quad 0 \leq t \leq T \\ dS_t &= S_{t-} d\widetilde{H}_t \end{aligned}$$

where

$$\widetilde{H}_t = H_t + \frac{1}{2}\langle H^c \rangle_t + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x)\mu^H(\mathrm{d}s, \mathrm{d}x)$$

Note

$$\mathcal{E}(\widetilde{H})_t = \exp\left(\widetilde{H}_t - \frac{1}{2}\langle \widetilde{H}^c \rangle_t\right) \prod_{0 < s \leq t} (1 + \Delta\widetilde{H}_s) \exp(-\Delta\widetilde{H}_s)$$

Asset price positive only if $\Delta\widetilde{H} > -1$.

Martingale modeling

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Let $\mathcal{M}_{\text{loc}}(P)$ be the class of local martingales.

Assumption ($\mathbb{ES}$)

The process $\mathbb{1}_{\{x>1\}}\mathrm{e}^{x} * \nu$ has bounded variation.

Then

$$S = S_0\mathrm{e}^{H} \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (\mathrm{e}^{x} - 1 - h(x)) * \nu = 0. \qquad (3)$$

Throughout, we assume that P is an equivalent martingale measure for S.

By the *Fundamental Theorem of Asset Pricing*, the value of an option on S equals the *discounted expected payoff* under this martingale measure.

We assume *zero* interest rates.

Supremum and infimum processes

Let $X = (X_t)_{0 \le t \le T}$ be a stochastic process. Denote by

$$\overline{X}_t = \sup_{0 \le u \le t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \le u \le t} X_u$$

the supremum and infimum process of X respectively. Since the exponential function is monotone and increasing

$$\overline{S}_T = \sup_{0 \le t \le T} S_t = \sup_{0 \le t \le T} \left(S_0 \mathrm{e}^{H_t} \right) = S_0 \mathrm{e}^{\sup_{0 \le t \le T} H_t} = S_0 \mathrm{e}^{\overline{H}_T}. \tag{4}$$

Similarly

$$\underline{S}_T = S_0 \mathrm{e}^{\underline{H}_T}. \tag{5}$$

Valuation formulae – payoff functional

We want to price an option with payoff $\Phi(S_t, 0 \leq t \leq T)$, where Φ is a measurable, non-negative functional.

Separation of payoff function from the underlying process:

Example

Fixed strike lookback option

$$(\overline{S}_T - K)^+ = (S_0 \, \mathrm{e}^{\overline{H}_T} - K)^+ = \left(\mathrm{e}^{\overline{H}_T + \log S_0} - K\right)$$

1. The *payoff function* is an arbitrary function $f : \mathbb{R} \to \mathbb{R}_+$; for example $f(x) = (\mathrm{e}^x - K)^+$ or $f(x) = \mathbb{1}_{\{\mathrm{e}^x > B\}}$, for $K, B \in \mathbb{R}_+$.
2. The *underlying process* denoted by X, can be the log-asset price process or the supremum/infimum or an average of the log-asset price process (e.g. $X = H$ or $X = \overline{H}$).

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Valuation formulae

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Consider the option price as a function of S_0 or better of $s = -\log S_0$

X driving process ($X = H, \overline{H}, \underline{H}$, etc.)

$$\Rightarrow \qquad \Phi(S_0\,\mathrm{e}^{H_t}, 0 \le t \le T) = f(X_T - s)$$

Time-0 price of the option (assuming $r \equiv 0$)

$$\mathbb{V}_f(X; s) = E\big[\Phi(S_t, 0 \le t \le T)\big] = E[f(X_T - s)]$$

Valuation formulae based on Fourier and Laplace transforms

Carr and Madan (1999) plain vanilla options

Raible (2000) general payoffs, Lebesgue densities

Borovkov and Novikov (2002) plain vanilla and lookback options

In these approaches: Some sort of continuity assumption (payoff or random variable)

Valuation formulae – assumptions

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

M_{X_T} moment generating function of X_T

$g(x) = \mathrm{e}^{-Rx} f(x)$ (for some $R \in \mathbb{R}$) dampened payoff function

$L^1_{bc}(\mathbb{R})$ bounded, continuous functions in $L^1(\mathbb{R})$

Assumptions

(C1) $g \in L^1_{bc}(\mathbb{R})$

(C2) $M_{X_T}(R)$ *exists*

(C3) $\widehat{g} \in L^1(\mathbb{R})$

Valuation formulae

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Theorem

Assume that (C1)–(C3) are in force. Then, the price $\mathbb{V}_f(X;s)$ of an option on $S=(S_t)_{0\le t\le T}$ with payoff $f(X_T)$ is given by

$$\mathbb{V}_f(X;s)=\frac{\mathrm{e}^{-Rs}}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{ius}\varphi_{X_T}(-u-iR)\widehat{f}(u+iR)\mathrm{d}u, \tag{6}$$

where φ_{X_T} denotes the extended characteristic function of X_T and $\widehat{f}$ denotes the Fourier transform of f.

Proof

$$\mathbb{V}_f(X;s)=\int_\Omega f(X_T-s)dP=e^{-Rs}\int_{\mathbb{R}}\mathrm{e}^{Rx}g(x-s)P_{X_T}(\mathrm{d}x). \tag{7}$$

cont. next page

Proof (cont.)

Under assumption (C1), $g \in L^1(\mathbb{R})$ and $\widehat{g}$ is well-defined. With (C3) $\widehat{g} \in L^1_{bc}(\mathbb{R})$.

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-ixu} \widehat{g}(u) \mathrm{d}u. \tag{8}$$

Returning to the valuation problem (7) we get

$$\begin{aligned} \mathbb{V}_f(X; s) &= e^{-Rs} \int_{\mathbb{R}} \mathrm{e}^{Rx} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-i(x-s)u} \widehat{g}(u) \mathrm{d}u \right) P_{X_T}(\mathrm{d}x) \\ &= \frac{\mathrm{e}^{-Rs}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ius} \left(\int_{\mathbb{R}} \mathrm{e}^{i(-u-iR)x} P_{X_T}(\mathrm{d}x) \right) \widehat{g}(u) \mathrm{d}u \\ &= \frac{\mathrm{e}^{-Rs}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ius} \varphi_{X_T}(-u - iR) \widehat{f}(u + iR) \mathrm{d}u. \end{aligned} \tag{9}$$

□

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Example (Asset-or-nothing digital)

Call payoff $f(x) = \mathrm{e}^x \mathbb{1}_{\{\mathrm{e}^x > B\}}$

$$\widehat{f}(u+iR) = -\frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (1, \infty)$$

Put payoff $f(x) = \mathrm{e}^x \mathbb{1}_{\{\mathrm{e}^x < B\}}$

$$\widehat{f}(u+iR) = \frac{B^{1+iu-R}}{1+iu-R}, \quad R \in I_1 = (-\infty, 1)$$

Example (Self-quanto option)

Call payoff $f(x) = \mathrm{e}^x (\mathrm{e}^x - K)^+$

$$\widehat{f}(u+iR) = \frac{K^{2+iu-R}}{(1+iu-R)(2+iu-R)}, \quad R \in I_1 = (2, \infty)$$

Non-path-dependent options

European option on an asset with price process $S_t = e^{H_t}$

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

$\longrightarrow \quad X_T \equiv H_T, \quad$ i.e. we need φ_{H_T}

Generalized hyperbolic model (GH model):

$$\varphi_{H_1}(u) = e^{iu\mu}\Big(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\Big)^{\lambda/2}\frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\,\big)}{K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)}$$

$$I_2 = (-\alpha-\beta, \alpha-\beta)$$

$$\varphi_{H_T}(u) = (\varphi_{H_1}(u))^T$$

similar: NIG, CGMY, Meixner

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Non-path-dependent options II

Stochastic volatility Lévy models: Carr, Geman, Madan, Yor (2003)
Eberlein, Kallsen, Kristen (2003)

Stochastic clock $Y_t = \int_0^t y_s ds \quad (y_s > 0)$

e.g. CIR process

$$dy_t = K(\eta - y_t)dt + \lambda y_t^{1/2} dW_t$$

Define for a pure jump Lévy process $X = (X_t)_{t \geq 0}$

$$H_t = X_{Y_t} \quad (0 \leq t \leq T)$$

Then

$$\varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu\varphi_{X_t}(-i)))^{iu}}$$

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Classification of option types

Lévy model $S_t = S_0 \mathrm{e}^{H_t}$

payoff	payoff function	distributional properties
$(S_T - K)^+$ call	$f(x) = (e^x - K)^+$	P_{H_T} usually has a density
$\mathbb{1}_{\{S_T > B\}}$ digital	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–
$\left(\overline{S}_T - K\right)^+$ lookback	$f(x) = (e^x - K)^+$	density of $P_{\overline{H}_T}$?
$\mathbb{1}_{\{\overline{S}_T > B\}}$ digital barrier = one touch	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–

Valuation formula for the last case

Payoff function f maybe discontinuous
P_{X_T} does not necessarily possess a Lebesgue density

Assumption

(D1) $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$

(D2) $M_{X_T}(R)$ *exists*

Theorem

Assume (D1)–(D2) then

$$\mathbb{V}_f(X; s) = \lim_{A \to \infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^{A} \mathrm{e}^{-ius} \varphi_{X_T}(u - iR) \widehat{f}(iR - u) \, \mathrm{d}u \qquad (10)$$

if $\mathbb{V}_f(X; \cdot)$ *is of bounded variation in a neighborhood of* s *and* $\mathbb{V}_f(X; \cdot)$ *is continuous at* s*.*

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Lévy processes

Let $L = (L_t)_{0 \leq t \leq T}$ be a *Lévy process* with triplet of local characteristics (b, c, λ), i.e. $B_t(\omega) = bt$, $C_t(\omega) = ct$, $\nu(\omega; dt, dx) = dt\lambda(\mathrm{d}x)$, λ Lévy measure.

Assumption $(\mathbb{EM})$

There exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} \mathrm{e}^{ux} \lambda(\mathrm{d}x) < \infty, \qquad \forall u \in [-M, M].$$

Using $(\mathbb{EM})$ and Theorems 25.3 and 25.17 in Sato (1999), we get that

$$E\left[\mathrm{e}^{uL_t}\right] < \infty, \quad E\left[\mathrm{e}^{u\overline{L}_t}\right] < \infty \quad \text{and} \quad E\left[\mathrm{e}^{u\underline{L}_t}\right] < \infty$$

for all $u \in [-M, M]$.

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

On the characteristic function of the supremum I

Lemma

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process that satisfies assumption $(\mathbb{EM})$. Then, the characteristic function $\varphi_{\overline{L}_t}$ of $\overline{L}_t$ has an analytic extension to the half plane $\{z \in \mathbb{C} : -M < \Im z < \infty\}$ and can be represented as a Fourier integral in the complex domain

$$\varphi_{\overline{L}_t}(z) = E\left[\mathrm{e}^{iz\overline{L}_t}\right] = \int_{\mathbb{R}} \mathrm{e}^{izx} P_{\overline{L}_t}(\mathrm{d}x).$$

Fluctuation theory for Lévy processes

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

Theorem
(Extension of Wiener–Hopf to the complex plane)

Let L be a Lévy process. The Laplace transform of $\overline{L}$ at an independent and exponentially distributed time θ, $\theta \sim \mathrm{Exp}(q)$, can be identified from the *Wiener–Hopf factorization* of L via

$$E\left[\mathrm{e}^{-\beta \overline{L}_\theta}\right] = \int_0^\infty q E[e^{-\beta \overline{L}_t}] e^{-qt}\,\mathrm{d}t = \frac{\kappa(q,0)}{\kappa(q,\beta)} \tag{11}$$

for $q > \alpha^*(M)$ and $\beta \in \{\beta \in \mathbb{C} | \mathcal{R}(\beta) > -M\}$ where $\kappa(q,\beta)$, is given by

$$\kappa(q,\beta) = k \exp\left(\int_0^\infty \int_0^\infty (e^{-t} - e^{-qt-\beta x}) \frac{1}{t} P_{L_t}(\mathrm{d}x)\,\mathrm{d}t\right). \tag{12}$$

On the characteristic function of the supremum II

Theorem

Let $L = (L_t)_{0 \le t \le T}$ be a Lévy process satisfying assumption $(\mathbb{EM})$. The Laplace transform of $\overline{L}_t$ at a fixed time t, $t \in [0, T]$, is given by

$$E\left[\mathrm{e}^{-\beta \overline{L}_t}\right] = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{\mathrm{e}^{t(Y+iv)}}{Y+iv} \frac{\kappa(Y+iv, 0)}{\kappa(Y+iv, \beta)} \mathrm{d}v, \tag{13}$$

for $Y > \alpha^*(M)$ and $\beta \in \mathbb{C}$ with $\Re\beta \in (-M, \infty)$.

Proof.

From (11) we get

$$\int_0^\infty \mathrm{e}^{-qt} E[\mathrm{e}^{-\beta \overline{L}_t}]\,\mathrm{d}t = \frac{1}{q} \frac{\kappa(q, 0)}{\kappa(q, \beta)}. \tag{14}$$

Invert the Laplace transform. □

Remark

Note that $\beta = -iz$ provides the characteristic function.

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

Application to lookback options

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Fixed strike lookback call: $(\overline{S}_T - K)^+$ (analogous for lookback put).

Combining the results, we get

$$\mathbb{C}_T(\overline{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi_{\overline{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} \mathrm{d}u \qquad (15)$$

where

$$\varphi_{\overline{L}_T}(-u - iR) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{\mathrm{e}^{T(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} \mathrm{d}v \qquad (16)$$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

- The floating strike lookback option, $(\overline{S}_T - S_T)^+$, is treated by a *duality* formula (Eb., Papapantoleon (2005)).

One-touch options

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

One-touch call option: $\mathbb{1}_{\{\overline{S}_T > B\}}$.

Driving Lévy process L is assumed to have infinite variation or has infinite activity and is regular upwards. L satisfies assumption $(\mathbb{EM})$, then

$$\begin{aligned} \mathbb{DC}_T(\overline{S}; B) &= \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{A} S_0^{R+iu} \varphi_{\overline{L}_T}(u - iR) \frac{B^{-R-iu}}{R+iu} \mathrm{d}u \qquad (17) \\ &= P(\overline{L}_T > \log(B/S_0)) \end{aligned}$$

for $R \in (0, M)$, $\varphi = \alpha^*(M)$ and

$$\varphi_{\overline{L}_T}(u - iR) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} \frac{\mathrm{e}^{T(Y+iv)}}{Y+iv} \frac{\kappa(Y+iv, 0)}{\kappa(Y+iv, -R-iu)} \mathrm{d}v. \qquad (18)$$

Equity default swap (EDS)

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

- Fixed premium exchanged for payment at “default”
- default: drop of stock price by 30 % or 50 % of $S_0 \to$ first passage time
- fixed leg pays premium $\mathcal{K}$ at times $T_1, \dots, T_N$, if $T_i \leq \tau_B$
- if $\tau_B \leq T$: protection payment C, paid at time τ_B
- premium of the EDS chosen such that initial value equals 0; hence

$$\mathcal{K} = \frac{CE\left[\mathrm{e}^{-r\tau_B}\mathbb{1}_{\{\tau_B \leq T\}}\right]}{\sum_{i=1}^{N} E\left[\mathrm{e}^{-rT_i}\mathbb{1}_{\{\tau_B > T_i\}}\right]}. \tag{19}$$

- Calculations similar to touch options, since $\mathbb{1}_{\{\tau_B \leq T\}} = \mathbb{1}_{\{\underline{S}_T \leq B\}}$.

Basic interest rates

$B(t,T)$: price at time $t \in [0,T]$ of a default-free zero coupon bond with maturity $T \in [0,T^*]$ $\quad (B(T,T) = 1)$

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$ as of time $t \le T$ $\quad$ (δ-forward Libor rate)

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities $T < U$

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

Dynamics of the forward rates

(Eb–Raible (1999), Eb–Özkan (2003), Eb–Jacod–Raible (2005), Eb–Kluge (2006))

$$df(t,T) = \alpha(t,T)\,dt - \sigma(t,T)\,dL_t \qquad (0 \le t \le T \le T^*)$$

$\alpha(t,T)$ and $\sigma(t,T)$ satisfy measurability and boundedness conditions and $\alpha(s,T) = \sigma(s,T) = 0$ for $s > T$

Define $\quad A(s,T) = \int_{s\wedge T}^{T} \alpha(s,u)\,du \quad$ and $\quad \Sigma(s,T) = \int_{s\wedge T}^{T} \sigma(s,u)\,du$

Assume $0 \le \Sigma^i(s,T) \le M \qquad (1 \le i \le d)$

For most purposes we can consider deterministic α and σ

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Implications

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Savings account and default-free zero coupon bond prices are given by

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t A(s,T)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \quad \text{and}$$

$$B(t,T) = B(0,T) B_t \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right).$$

If we choose $A(s,T) = \theta_s(\Sigma(s,T))$, then bond prices, discounted by the savings account, are martingales.

In case $d = 1$, the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).

Key tool

$L = (L^1, \ldots, L^d)$ $\qquad$ d-dimensional time-inhomogeneous Lévy process

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\,\mathrm{d}s \qquad \text{where}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

in case L is a (time-homogeneous) Lévy process, $\theta_s = \theta$ is the cumulant (log-moment generating function) of L_1.

Proposition Eberlein, Raible (1999)

Suppose $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\mathcal{R}(f^i(x))| \leq M$ for all $i \in \{1, \ldots, d\}$ and $x \in \mathbb{R}_+$, then

$$\mathbb{E}\left[\exp\left(\int_0^t f(s)dL_s\right)\right] = \exp\left(\int_0^t \theta_s(f(s))ds\right)$$

Take $\quad f(s) = \sum(s, T)$ for some $T \in [0, T^*]$

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Pricing of European options

$$B(t,T) = B(0,T)\exp\left[\int_0^t (r(s) + \theta_s(\Sigma(s,T)))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\mathrm{d}L_s\right]$$

where $r(t) = f(t,t)$ short rate

$V(0,t,T,w)$ time-0-price of a European option with maturity t and payoff $w(B(t,T),K)$

$$V(0,t,T,w) = \mathbb{E}_{\mathbb{P}^*}[B_t^{-1} w(B(t,T),K)]$$

Volatility structures

$$\Sigma(t,T) = \frac{\widehat{\sigma}}{a}(1 - \exp(-a(T-t))) \qquad \text{(Vasiček)}$$

$$\Sigma(t,T) = \widehat{\sigma}(T-t) \qquad \text{(Ho–Lee)}$$

Fast algorithms for Caps, Floors, Swaptions, Digitals, Range options

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Forward measure associated with data $T \leq T^*$

Density $\frac{d\mathbb{P}_T}{d\mathbb{P}^*} = \frac{1}{B_T B(0,T)}$ or $\mathbb{E}_{\mathbb{P}^*}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}^*} \mid \mathcal{F}_t\right] = \frac{B(t,T)}{B_t B(0,T)}$

For the case of the Lévy term structure model this equals

$$\exp\left(\int_0^t \Sigma(s,T)\,dL_s - \int_0^t \theta_s(\Sigma(s,T))\,ds\right)$$

Compensator of μ^L under $\mathbb{P}_T$: $\nu^T(dt,dx) = e^{\langle \Sigma(t,T),x\rangle}\nu(dt,dx)$

Standard Brownian motion under $\mathbb{P}_T$: $W_t^T = W_t - \int_0^t c_s^{1/2}\Sigma(s,T)\,ds$

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Pricing formula for caps

(Eberlein, Kluge (2006))

$$w(B(t,T),K) = (B(t,T)-K)^+$$

Call with strike K and maturity t on a bond that matures at T

$$\begin{aligned} C(0,t,T,K) &= \mathbb{E}_{\mathbb{P}^*}[B_t^{-1}(B(t,T)-K)^+] \\ &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T)-K)^+] \end{aligned}$$

Assume $X = \int_0^t (\Sigma(s,T) - \Sigma(s,t))dL_s$ has a Lebesgue density, then

$$\begin{aligned} C(0,t,T,K) &= \frac{1}{2\pi} KB(0,t)\exp(R\xi) \\ &\quad \times \int_{-\infty}^{\infty} e^{iu\xi}(R+iu)^{-1}(R+1+iu)^{-1} M_t^X(-R-iu)du \end{aligned}$$

where ξ is a constant and $R < -1$.

Analogous for the corresponding put and for swaptions

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

Basic interest rates

$B(t,T)$: price at time $t \in [0, T]$ of a default-free zero coupon bond

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities T and U

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

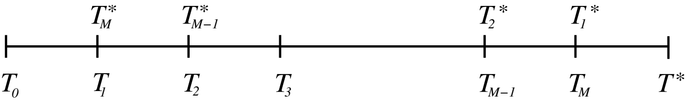
$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

The Lévy Libor model

(Eb–Özkan (2005))

Tenor structure $T_0 < T_1 < \cdots < T_M < T_{M+1} = T^*$
with $T_{i+1} - T_i = \delta$, set $T_i^* = T^* - i\delta$ for $i = 1, \ldots, M$



Assumptions

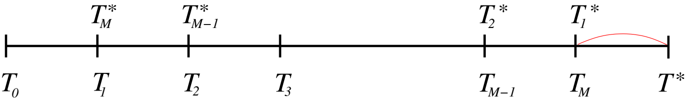
(LR.1): For any maturity T_i there is a bounded deterministic function $\lambda(\cdot, T_i)$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$.

(LR.2): We assume a strictly decreasing and strictly positive initial term structure $B(0, T)$ $(T \in]0, T^*])$. Consequently the initial term structure of forward Libor rates is given by

$$L(0, T) = \frac{1}{\delta}\left(\frac{B(0, T)}{B(0, T + \delta)} - 1\right)$$

Backward Induction

Given a stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0 \le t \le T^*})$



We postulate that under $\mathbb{P}_{T^*}$

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*} \right)$$

where $$L_t^{T^*} = \int_0^t b_s ds + \int_0^t c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T^*,L})(ds, dx)$$

is a non-homogeneous Lévy process with random measure of jumps μ^L and $\mathbb{P}_{T^*}$-compensator $\nu^{T^*,L}(ds, dx) = F_s(dx)ds$

Pricing of caps and floors

Time-T_j-payoff of a cap settled in arrears

$$N\delta(L(T_{j-1},T_{j-1})-K)^+$$

N notional amount (set $N=1$)

K strike rate

Time-t value

$$\begin{aligned} C_t &= \sum_{j=1}^n \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_t}{B_{T_j}}\,\delta(L(T_{j-1},T_{j-1})-K)^+ \,|\, \mathcal{F}_t\right] \\ &= \sum_{j=1}^n B(t,T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\left[\delta(L(T_{j-1},T_{j-1})-K)^+ \,|\, \mathcal{F}_t\right] \end{aligned}$$

Analogous for floor

$$N\delta(K-L(T_{j-1},T_{j-1}))^+$$

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

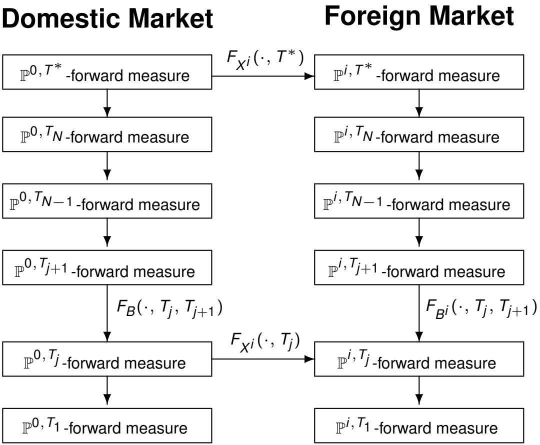
References

Extensions of the basic Lévy market model

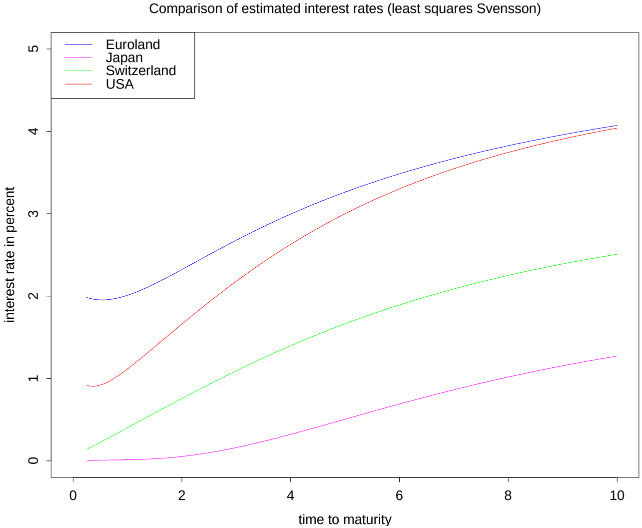
Lévy market model
(Eb–Özkan (2005))

- Multi-currency setting
  (Eb–Koval (2006))
- Credit risk model
  (Eb–Kluge–Schönbucher (2006))
- Swap rate model
  (Eb–Liinev (2006))
- Duality principle
  (Eb–Kluge–Papapantoleon (2006))

Cross-currency Lévy market model



Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.



Termstructure, February 17, 2004

The model

Valuation

Examples of payoff functions

Lévy processes

Exotic options

Interest rate derivatives

References

Pricing cross-currency derivatives

Foreign forward caps and floors

$$\delta X[L^i(T_{j-1},T_{j-1}) - K^i]^+$$

Time-0-value of a foreign T_N-maturity cap

$$\mathrm{FC}^i(0,T_N) = \delta \sum_{j=1}^{N+1} B^i(0,T_j)\mathbb{E}_{\mathbb{P}^{i,T_j}}\left[\left(L^i(T_{j-1},T_{j-1}) - K^i\right)^+\right]$$

Alternatively if we define $\widetilde{K}^i = 1 + \delta K^i$ (forward process approach)

$$\begin{aligned}\mathrm{FC}^i(0,T_N) &= \sum_{j=1}^{N+1} B^i(0,T_j)\mathbb{E}_{\mathbb{P}^{i,T_j}}\left[\left(1 + \delta L^i(T_{j-1},T_{j-1}) - \widetilde{K}^i\right)^+\right], \\ &= \sum_{j=1}^{N+1} C^i(0,T_j,\widetilde{K}^i)\end{aligned}$$

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

Pricing cross-currency derivatives (cont.)

The model
Valuation
Examples of payoff functions
Lévy processes
Exotic options
Interest rate derivatives
References

A quanto caplet with strike K^i, which expires at time T_{j-1}, pays at time T_j

$$QCpl^i(T_j, T_j, K^i) = \delta \overline{X}^i (L^i(T_{j-1}, T_{j-1}) - K^i)^+$$

where $\overline{X}^i$ is the preassigned foreign exchange rate

Time-0-value

$$\begin{aligned} QCpl^i(0, T_j, K^i) &= B^0(0, T_j)\, \mathbb{E}_{\mathbb{P}^{0,T_j}}[\delta \overline{X}^i (L^i(T_{j-1}, T_{j-1}) - K^i)^+] \\ &= B^0(0, T_j) \overline{X}^i\, \mathbb{E}_{\mathbb{P}^{0,T_j}}[(1 + \delta L^i(T_{j-1}, T_{j-1}) - (1 + \delta K^i))^+] \end{aligned}$$

(forward process approach)

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