Asymptotic behaviour of densities related to Stable processes .

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1 Introduction and results

Let X be a strictly stable Lévy process of index $\alpha \in (0, 2)$ which has positive jumps and is not a subordinator; then its Lévy measure has density

$$\nu(x) = \begin{cases} c_+ x^{-\alpha-1}, & x > 0, \\ c_-|x|^{-\alpha-1}, & x < 0, \end{cases}$$

where $c_+ > 0, c_- \geq 0$. For $x > 0$ put

$$\tau_x = \inf\{t : X_t > x\}.$$

Using the scaling property, with $S_t = \sup_{s \leq t} X_s$ we have

$$P(\tau_x > t) = P(S_t \leq x) = P(S_1 \leq xt^{-\eta}),$$

where $\eta = 1/\alpha$. It is known that S_1 has a continuous density function m say, so τ_x has a continuous density function given by

$$h_x(t) = \eta x t^{-\eta-1} m(xt^{-\eta}).$$

QUESTION How does h behave for fixed $x > 0$ as $t \to \infty$?

Clearly we can answer this question if we know how $m(y)$ behaves as $y \downarrow 0$.

Bingham (1973) showed that

$$P(S_1 \leq y) = \int_0^y m(u)du \backsim \frac{Ay^{\alpha\rho}}{\alpha\rho} \text{ as } y \downarrow 0, \qquad (1)$$

where $\rho = P(X_1 > 0)$ and A is a known constant, so the obvious conjecture is that

$$m(y) \backsim Ay^{\alpha\rho - 1} \text{ as } y \downarrow 0. \qquad (2)$$

Now write $p_t(x)$ for the density function of $\theta_t^{(t)}$, where $(\theta_s^{(t)}, 0 \leq s \leq t)$ denotes the corresponding stable meander of length t; informally this is X conditioned to stay positive up to time t. The meander inherits the scaling property, so $p_t(x) = t^{-\eta}p(xt^{-\eta})$, where we write p for p_1.

THEOREM There is a constant C such that

$$p(x) \backsim Cx^{\alpha\rho} \text{ as } x \downarrow 0,$$

(2) holds, and for x fixed, $h_x(t) \backsim \eta A x^{\alpha\rho} t^{-\alpha-1}$ as $t \to \infty$.

2 Some connections

Another quantity that appears in our analysis is $g(t,x)$, the bivariate density of the renewal measure of the ladder process (T,H) of X, i.e.

$$\begin{aligned} G(dt,dx) &= \int_0^\infty P(T_s \in dt, H_s \in dx) \\ &= g(t,x)dtdx. \end{aligned}$$

We also have

$$\begin{aligned} g(t,x)dx &= n^*(\epsilon_t \in dx, \zeta > t) \\ &= ct^{\rho-1} n^*(\epsilon_t \in dx | \zeta > t) = ct^{\rho-1} p_t(x)dx, \end{aligned}$$

where $ct^{\rho-1}$is the value of $\overline{\Pi^*}(t) = n^*(\zeta > t)$, the tail of the Lévy measure of T^*. We can connect m with p and $\overline{\Pi}(t) = ct^{-\rho}$ through g, the result being

$$\begin{aligned} m(x) &= \frac{\sin \rho\pi}{\pi} \int_0^1 \frac{p_s(x)ds}{s^{1-\rho}(1-s)^\rho} \\ &= \frac{\sin \rho\pi}{\pi} \int_0^1 \frac{s^{-\eta} p(x)ds}{s^{1-\rho}(1-s)^\rho}. \end{aligned} \tag{3}$$

Finally we write $\kappa(x)$ for $\kappa_1(x)$, where $\kappa_t(x) = ct^{\rho-1}p_t(x)$ is the density of the measure $n^*(\epsilon_t \in dx, \zeta > t)$.

3 Proofs

The existence and continuity of the densities comes from the fact that X_t has a bounded and continuous density f_t by the following argument: write

$$Q_y(X_t \in dx) = P_y(X_t \in dx, \sigma_0 > t)$$

for the transition function of the process killed on leaving $[0, \infty)$. Then $Q_y\ (X_t \in \cdot)$ has a continuous and bounded density function, given by

$$q_t(y, x) = f_t(x - y) - \tilde{f}_t(y, x),$$

where

$$\tilde{f}_t(y, x) = \int_0^t \int_{-\infty}^0 P_y(\sigma_0 \in ds, X_{\sigma_0} \in dz) f_{t-s}(x-z).$$

It follows that

$$\begin{aligned} n^*(\epsilon_t &\in dx, \zeta > t) \\ &= \int_0^\infty n^*(\epsilon_{t/2} \in dy, \zeta > t/2) q_{t/2}(y,x) dx, \end{aligned}$$

and the existence, continuity, and boundedness of $\kappa_t(\cdot)$, and hence of $p_t(\cdot)$, follow.

Putting $t = 2$ and rewriting in terms of κ this says

$$\kappa_2(x) = \int_0^\infty \kappa(y) q_1(y,x) dy.$$

But duality gives $q_1(y,x) = q_1^*(x,y)$, so

$$\begin{aligned} \kappa_2(x) &= \int_0^\infty \kappa(y) q_1^*(x,y) dy \leq c \int_0^\infty q_1^*(x,y) dy \\ &= cP_{-x}(S_1 < 0) = cP_0(S_1 < x) \backsim cx^{\alpha\rho}. \end{aligned}$$

Then we recall that the law of "$-X$ starting at $x > 0$ and conditioned to stay positive" has, at time 1, density

$$\begin{aligned} p_1^{*\uparrow}(x,y) \quad &:= q_1^*(x,y) \frac{y^{\alpha\rho}}{x^{\alpha\rho}} \\ &\to p_1^{*\uparrow}(0,y) \text{ as } x \downarrow 0. \end{aligned}$$

Thus, if we write $g(y) = \kappa(y)y^{-\alpha\rho}$, which is bounded, we see that in the obvious notation,

$$\begin{aligned} \kappa_2(x)x^{-\alpha\rho} &= \int_0^\infty y^{-\alpha\rho}\kappa(y)y^{\alpha\rho}q_1^*(x,y)x^{-\alpha\rho}dy \\ &= \int_0^\infty g(y)p_1^{*\uparrow}(x,y)dy \\ &\to \int_0^\infty g(y)p_1^{*\uparrow}(0,y)dy < \infty. \end{aligned}$$

This proves the first statement, and also allows us to see that $p_1^{\uparrow}(x) \backsim cx^{\alpha}$ as $x \downarrow 0$.

We deduce the behaviour of p from (3): start by putting $s^{-\eta} = z$ and then $zx = y$, to get

$$\begin{aligned} m(x) &= \frac{\alpha \sin \rho\pi}{\pi} \int_1^\infty \frac{p(xz)}{(z^\alpha - 1)^\rho} dz \\ &= \frac{\alpha x^{\alpha\rho - 1} \sin \rho\pi}{\pi} \int_x^\infty \frac{p(y)}{(y^\alpha - x^\alpha)^\rho} dy. \end{aligned}$$

Since we have seen that p is bounded,

$$\begin{aligned}\lim_{\delta\downarrow 0}\int_x^{(1+\delta)x}\frac{p(y)dy}{(y^\alpha - x^\alpha)^\rho} &\le c\lim_{\delta\downarrow 0}\int_x^{(1+\delta)x}\frac{dy}{(y^\alpha - x^\alpha)^\rho}\\ &= cx^{1-\alpha\rho}\lim_{\delta\downarrow 0}\int_1^{1+\delta}\frac{dz}{(z^\alpha - 1)^\rho} = 0,\end{aligned}$$

as the integral is finite because $\rho < 1$. Moreover on $((1+\delta)x, \infty)$, $(y^\alpha - x^\alpha)^{-\rho} \le y^{-\alpha\rho}(1-(1+\delta)^{-\alpha})^{-\rho}$, and, since we know $y^{-\alpha\rho}p(y)$ is integrable on $(0, \infty)$, for any $\delta > 0$ dominated convergence gives

$$\int_{(1+\delta)x}^{\infty}\frac{p(y)}{(y^\alpha - x^\alpha)^\rho}dy \to \int_0^\infty y^{-\alpha\rho}p(y)dy < \infty.$$

Thus $\lim_{x\downarrow 0} x^{1-\alpha\rho}m(x) \in (0, \infty)$, and since (1) holds, the limit must be A.

4 Large time problem

Using somewhat similar arguments, we find that

$$m(x) \backsim \rho p(x) \backsim Bx^{-(\alpha+1)} \text{ as } x \to \infty,$$

where B is another known constant such that $P(X_1 > x) \backsim B\alpha^{-1}x^{-\alpha}$.

A consequence is that for fixed x

$$h_x(t) \to B\eta/x^{\alpha} \text{ as } t \downarrow 0.$$

As well as the other connections, our proof uses the following identity, which is due to Larbi Alili:

$$xg(t,x) = \int\limits_{u=0}^{t} \int\limits_{z=0}^{x} g(u,z) \frac{x-z}{t-u} f_{t-u}(x-z) du dz.$$