# A continuum tree-valued Markov process

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# Outline

1. Discrete case
2. Continuum random trees
3. Continuum-tree valued process

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# Discrete case: Galton-Watson (GW) tree

- Offspring distribution: $\mathcal{P}(\mathrm{e}^{-\theta_0})$. G-W tree: $GW_{\theta_0}$. It is
  - sub-critical if $\theta_0 > 0$,
  - critical if $\theta_0 = 0$,
  - super-critical if $\theta_0 < 0$ (and $\mathbb{P}(\text{Card }(GW_{\theta_0}) = +\infty) \in (0, 1]$).
- Pruning of GW tree.
  - Add indep. exponential random variables $\tau$ (mean=1) on each branch.
  - Let $\theta > 0$. Cut all branches s.t. $\tau \leq \theta$.
  - The sub-tree (with the initial root) is a GW tree with offspring distribution $\mathcal{P}(\mathrm{e}^{-(\theta_0+\theta)})$: $GW_{\theta_0+\theta}$.

# Discrete tree-valued Markov process

- Decreasing tree-valued Markov process: $\theta \mapsto GW_{\theta_0+\theta}$.
- Consistency (or time reversal) allows to define $\theta \mapsto GW_\theta$ for $\theta \in \mathbb{R}$.
- Aldous and Pitman 1998: distribution of the explosion time

$$A = \inf\{\theta; \text{Card}\,(GW_\theta) < +\infty\} \quad (< 0)$$

and the distribution of $GW_{A+}$.

# Continuous branching (CB) process

- $Z'_n$: number of people at the $n$-th generation.
- Lamperti 1967: $(Z'_n, n \in \mathbb{N})$ properly rescaled converges to a CB process $Z = (Z_t, t \in \mathbb{R}_+)$ with $Z_t$ the "mass" of a population at time $t$ whose individuals are of infinitesimal mass and have infinitesimal lifetime.

$Z$ is Markov and

$$\mathbb{E}^{\psi}\left[\mathrm{e}^{-\lambda Z_t} | Z_0 = x\right] = \mathrm{e}^{-xu(t,\lambda)},$$

where $\partial_t u + \psi(u) = 0$ and $u(0, \lambda) = \lambda$, and

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(dr) \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r \leq 1\}}\right)$$

($\beta \geq 0$ and $\pi$ Lévy measure).

# Continuous branching (CB) process

We have

$$\mathbb{E}^{\psi}[Z_t|Z_0 = x] = x\,\mathrm{e}^{-\psi'(0)} \quad \text{and} \quad Z_t \xrightarrow[t\to\infty]{a.s.} Z_\infty \in \{0, +\infty\}.$$

We set $\sigma = \int_0^\infty Z_s\, ds$ the total population mass. $Z$ is

- sub-critical if $\psi'(0) > 0$: a.s. $Z_\infty = 0$ and $\sigma < +\infty$,
- critical if $\psi'(0) = 0$: a.s. $Z_\infty = 0$ and $\sigma < +\infty$,
- super-critical if $\psi'(0) < 0$: $\{\sigma < +\infty\} = \{Z_\infty = 0\}$ and $\mathbb{P}^{\psi}(Z_\infty = 0|Z_0 = x) = \mathrm{e}^{-xq_0}$.

Excursion (canonical) measure: $\mathbb{N}^{\psi}[Z \in A] = \lim_{x\to 0} \dfrac{1}{x}\, \mathbb{P}^{\psi}(Z \in A|Z_0 = x)$. It describes the descendants of an infinitesimal individual.

# Representation of the Lévy tree (critical or sub-critical case)

Aldous 1991, Duquesne and Le Gall 2002.
Under $\mathbb{N}^\psi$, the tree $T$ has the following property:

- $H$ has "density" $\mathrm{e}^{-\psi'(0)a}\,\mathbf{1}_{\{a>0\}}\,da$.
- $(h_i, Z^i)$ is a Poisson point measure with intensity

$$2\beta\mathbf{1}_{[0,H]}(h)\,dh\,\mathbb{N}^\psi[dZ],$$

- $(h_j, Z^j, \ell_j)$ is a Poisson point measure with intensity

$$\mathbf{1}_{[0,H]}(h)\,dh\,\ell\pi(d\ell)\,\mathbb{P}^\psi(dZ|Z_0=\ell).$$

- Cond. on $H$, the two Poisson point measures are independent.

# Pruning the Lévy tree (critical or sub-critical case)

Abraham, D. and Voisin (preprint 2008).
Let $T$ be a Lévy tree associated to $\psi$. Let $\theta > 0$.

- Add indep. exponential random variables $\tau$ (mean= 1/size of the node) on each node.
- Put marks on branches (i.e. skeleton) with rate $2\beta\theta$.
- Cut all nodes s.t. $\tau \leq \theta$ and all marks to get a sub-tree $T_\theta$ (with the initial root).

**Theorem**

*The sub-tree, $T_\theta$ is a Lévy tree associated to $\psi_\theta$:*

$$\psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta).$$

$(T_\theta, \theta \in \mathbb{R}_+)$ *is a decreasing continuum tree-valued Markov process.*

# Super-critical case

We set $\Theta = \{\theta \in \mathbb{R};\ \ \psi_\theta \text{ is well defined}\} \supset \mathbb{R}_+$.
Examples:

- $\psi(\lambda) = \lambda^2$: $\Theta = \mathbb{R}$.
- $\alpha \in (1, 2)$. $\psi(\lambda) = \lambda^\alpha$: $\Theta = \mathbb{R}_+$.
- $\psi(\lambda) = (\lambda + \mathrm{e}^{-1}) \log(\lambda + \mathrm{e}^{-1}) + \mathrm{e}^{-1}$: $\Theta = [-\mathrm{e}^{-1}, +\infty)$.
  (Notice $\psi_{-\mathrm{e}^{-1}}(\lambda) = \lambda \log(\lambda)$.)

**Theorem (Girsanov transformation)**

*Let $\psi$ be critical or sub-critical. Let $\theta \in \Theta$.*

$$\frac{d\mathbb{P}^{\psi_\theta}_{|\mathcal{F}_t}}{d\mathbb{P}^{\psi}_{|\mathcal{F}_t}} = \exp\left(\theta Z_0 - \theta Z_t - \psi(\theta) \int_0^t Z_s \, ds\right),$$

*where $\mathcal{F}_t$ is the $\sigma$-field generated by the tree up to level $t$ ($\supset \sigma(Z_s, s \in [0, t])$).*

# Continuum-tree valued process

Assume $\psi$ is critical.

- Using Girsanov transformation, extend the pruning to the super-critical case.
- $(\mathcal{T}_\theta, \theta \in \Theta)$ is a decreasing continuum tree-valued Markov process.
- Set $\sigma_\theta = \int_0^\infty Z_t^\theta \, dt$ the total mass of $\mathcal{T}_\theta$. Define the explosion time

$$A = \inf\{\theta; \sigma_\theta < +\infty\} \quad (< 0).$$

### Theorem

*Let* $\theta < 0$. $\mathbb{N}^\psi[A \geq \theta] = \bar{\theta} - \theta$, *where* $\bar{\theta} > 0$ *is s.t.* $\psi(\theta) = \psi(\bar{\theta})$.

# Tree at the explosion time

Assume $\psi$ is critical.

- Define $T^*$ as $T_0$ but for $H = +\infty$ (intuitively $T^* \sim T_0$ conditionally on non extinction).
- Use the pruning procedure to define $(T^*_\theta, \theta \geq 0)$.

## Theorem

- *Cond. on* $\{A = \theta\}$, $T_{A+}$ *is distributed as* $T^*_{\bar{\theta}}$.
- $(T_{A+\theta}, \theta \geq 0)$ *is distributed as* $(T^*_{A'+\theta}, \theta \geq 0)$ *where* $A'$ *is independent of* $T^*$ *and distributed as* $\bar{A}$.

# The quadratic case: $\psi(\lambda) = \lambda^2/2$

- $\mathbb{N}^\psi[A \geq \theta] = \bar{\theta} - \theta = 2|\theta|$ for $\theta < 0$.
- For $\theta < 0$,
$$\mathbb{N}^\psi[\mathrm{e}^{-\lambda\sigma_A} | A = \theta] = \frac{\sqrt{\theta^2}}{\sqrt{2\lambda + \theta^2}},$$
that is, cond. on $\{A = \theta\}$, $\sigma_A$ is gamma $(\theta^2/2, 1/2)$.
- Set $\sigma_\theta^*$ the total mass of $T_\theta^*$. We have:
  - $(\sigma_\theta^*, \theta \geq 0)$ is Markov.
  - For $\theta > 0$ and $q > 0$,
$$\mathbb{E}^\psi[\mathrm{e}^{-\lambda\sigma_\theta^* - \kappa\sigma_{\theta+q}^*}] = \frac{\sqrt{\theta^2}}{\sqrt{2\lambda + \theta^2}} \frac{\sqrt{q^2} + \sqrt{2\lambda + \theta^2}}{\sqrt{2\kappa + \left(\sqrt{q^2} + \sqrt{2\lambda + \theta^2}\right)^2}}.$$
- Conclusion: $(1/\sigma_\theta^*, \theta \geq 0)$ is a stable subordinator with index $1/2$ (first passage process of a Brownian motion). See also Aldous and Pitman 1998 on additive coalescent and Bravo 2008 on the size of a tagged fragment for the height Brownian fragmentation.