Joint work with P. Abry, P. Chainais and V. Pipiras

Multifractal random Walks as fractional Wiener integrals

L. Coutin
MAP5
Université Paris Descartes

July 22, 2009

Motivations

Random multifractal measures or processes have been used to model natural and made man phenomena in

- ranging from turbulence in hydrodynamics,
- DNA sequences,
- teletraffic in Internet.

Two complementary ways of description

- Hausdorff dimension of singularity exponents of multifractals,
- scaling properties of moments of multifractals,

$$\zeta(p) := \frac{\log \mathbb{E}(|X(t+\tau) - X(t)|^p)}{\log \tau}.$$

Random multifractal measures

Random multifractal measures denoted by X

- multiplicative (binomial) cascades (Mandelbrot 1974, Kahane and Peyriere 1976),
- compound Poisson cascades (Barral and Mandelbrot 2002),
- (log)-infinitely divisble multifractals measures (Bacry and Muzy 2003)
- non-scale invariant infinitely divisible cascades (Abry, Chainais and Reidi 2005)

Random multifractal processes

Random multifractal processes Z can be obtained

- by subordination

$$Z(t) = Y(X([0, t])), \quad t \in [0, T]$$

where X is a random multifractal measure and an independent self-similar process Y with stationary increments, Mandelbrot 1999,

Random multifractal processes

Random multifractal processes Z can be obtained

- by subordination

$$Z(t) = Y(X([0,t])), \quad t \in [0,T]$$

  where X is a random multifractal measure and an independent self-similar process Y with stationary increments, Mandelbrot 1999,

- or by integration

$$Z(t) = \int_0^t Q(u) dY(u), t \in [0,T]$$

  where "Q" is a "infinitely divisible cascading noise"', i.e. the density of a random multifractal measure X .

Random multifractal processes

Random multifractal processes Z can be obtained

- by subordination

$$Z(t) = Y(X([0,t])), \quad t \in [0,T]$$

  where X is a random multifractal measure and an independent self-similar process Y with stationary increments, Mandelbrot 1999,
- or by integration

$$Z(t) = \int_0^t Q(u)dY(u), t \in [0,T]$$

  where "Q" is a "infinitely divisible cascading noise"', i.e. the density of a random multifractal measure X .
- The process Z when Y is fractional Brownian motion is introduced in Bacry and Muzy 2003.

Infinitely divisible cascading noise (IDC)

Bacry and Muzy (2003) or Chainais Abry and Reidi (2005)

- ψ is an exponent function of a infinitely divisible r.v. θ and , ρ is defined as $\mathbb{E}(\exp q\theta) = \exp -\rho(q)$.
  ρ is concave and $\rho(0) = 0$. Latter $\rho(2) > -\infty$ (no stable distribution)
- m is a control mesure on $\mathbb{P} = \mathbb{R} \times \mathbb{R}^+$

M is an **infinitely divisible cascading scattered random measure** on $\mathbb{P}$ i.e.

- if $A_1, \ldots, A_n$ are mutually disjoint borelian sets, then $M(A_1), \ldots, M(A_n)$ are independent random variables,
- $\mathbb{E}(\exp iM(A_1)) = \exp \psi(q) m(A_1)$.

Infinitely divisible cascading noise (IDC)

- Definition

  An IDC noise is $(Q_r(t))_{t\geq 0}$, where

$$Q_r(t) = \frac{\exp M(C_r(t))}{\exp -\rho(1) m(C_r(t))}$$

and $C_r(t) = \{(t', r') : \quad r \leq r' \leq 1, \quad t - \frac{r'}{2} \leq t' < t + \frac{r'}{2}\}$

  - $\varphi(q) = \rho(q) - q\rho(1)$.
  - For fixed r, Q_r is a stationnary process.
  - For fixed t, $(Q_{1/R}(t))_R$ is a positive martingale.

Infinitely divisible cascading noise (IDC)

Following Bacry and Muzy 2003, we consider only two cases

- **Exact scale invariance**

$$dm(t,r) = dt(\frac{cdr}{r^2} + c\delta_1 dr)\mathbf{1}_{]0,1]}(r).$$

- **Scale invariance**

$$dm(t,r) = dt(\frac{cdr}{r^2})\mathbf{1}_{]0,1]}(r).$$

Lemma

- *In the case of exact scale invariance,* $(Q_{r\lambda}(\lambda t))_{t\in\mathbb{R}} =^d e^{\Omega_\lambda}(Q_r(t))_{t\in\mathbb{R}}$ *with* $\mathbb{E}(e^{q\Omega_\lambda}) = \lambda^{c\varphi(q)}$.
- *In the case of scale invariance* $(\frac{Q_{r\lambda}(\lambda t)}{Q_\lambda(\lambda t)})_{t\in\mathbb{R}} =^d (Q_{r/\lambda}(t))_{t\in\mathbb{R}}$

Multifractal random measure

Following Bacry and Muzy 2003, $X_r([0,t]) = \int_0^t Q_r(u)du$.

Definition

The MRM measure X is defined as the limit measure of X_r when r goes to 0.

Note that $X(\{t\}) = 0$.

Proposition

- If $\exists q > 1,\ q + c\varphi(q) > 1$,
  then $\mathbb{E}(X([0,t])) = t$,
  $\mathbb{E}(|X([0,t])|^q)$ behaves as $t^{q+c\varphi(q)}$,
  and $\mathbb{E}(\sup_{t\in[0,1]} Q_r(t)^q)) < +\infty$.

Multifractal random measure

Following Bacry and Muzy 2003, $X_r([0,t]) = \int_0^t Q_r(u)du$.

Definition

The MRM measure X is defined as the limit measure of X_r when r goes to 0.

Note that $X(\{t\}) = 0$.

Proposition

- If $\exists q > 1,\ q + c\varphi(q) > 1$,
  then $\mathbb{E}(X([0,t])) = t$,
  $\mathbb{E}(|X([0,t])|^q)$ behaves as $t^{q+c\varphi(q)}$,
  and $\mathbb{E}(\sup_{t\in[0,1]} Q_r(t)^q)) < +\infty$.
- If for $q > 1$ $\mathbb{E}(|X([0,t])|^q) < \infty$ then $q + c\varphi(q) \geq 1$.

Fractional Wiener integral

Definition

Fractional Brownian motion is the unique Gaussian centered process B^H such that $\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$.

- **Proposition** Pipiras and Taqqu 2003
  For $H > 1/2$, The map defined on the steps functions $f(u) = \sum_{i=0}^{n} a_i \mathbf{1}_{[t_i, t_{i+1}[}(u), \quad u \in [0,1]$ by $\int_0^T f(u) dB^H(u) = \sum_{i=1}^{n} a_i [B^H(t_{i+1}) - B^H(t_i)]$ extends into an isometry from $\mathcal{H}^H$ into $L^2(\Omega)$

Fractional Wiener integral

Definition

Fractional Brownian motion is the unique Gaussian centered process B^H such that $\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$.

- **Proposition** Pipiras and Taqqu 2003
  For $H > 1/2$, The map defined on the steps functions $f(u) = \sum_{i=0}^{n} a_i \mathbf{1}_{[t_i,t_{i+1}[}(u), \quad u \in [0,1]$ by $\int_0^T f(u)dB^H(u) = \sum_{i=1}^{n} a_i[B^H(t_{i+1}) - B^H(t_i)]$ extends into an isometry from $\mathcal{H}^H$ into $L^2(\Omega)$
- where

$$\mathcal{H}^H = \{f \; : [0,T] \to \mathbb{R} | \int_{[0,T]^2} |f(u)||f(v)||u-v|^{2H-2} dudv < +\infty\}$$

endowed with $< f, g >_{\mathcal{H}^H} = \int_{[0,T]^2} f(u)f(v)|u-v|^{2H-2} dudv$ called **Fractional Wiener integral**.

Fractional Wiener integral of IDC noise

- Lemma
  For all $r \in]0,1]$, $Q_r \in \mathcal{H}^H$ *a.s.* .
  - Let B^H and M be independent and Z_r^H be

$$Z_r^H(t) = \int_0^t Q_r(u) dB^H(u), \quad t \in [0,T].$$

  - For fixed t, $(Z_{1/R}^H(t))_R$ is a martingale.
- Definition
  MRW process Z^H is the limit (if it exists) of Z_r^H.

Existence of MRW

- **Proposition** Bacry and Muzy 2003 If $2H + c\varphi(2) > 1$, then for all $t > 0$, $(Z_r^H(t))_r$ converges in L^2, when $r \to 0$.

Existence of MRW

- **Proposition** Bacry and Muzy 2003 If $2H + c\varphi(2) > 1$, then for all $t > 0$, $(Z_r^H(t))_r$ converges in L^2, when $r \to 0$.
- **Corollary** If $2H + c\varphi(2) > 1$, then $((Z_r^H(t), \quad t \in [0, T]))_r$ converges almost surely in $C([0, T], \mathbb{R})$.

Existence of MRW

- **Proposition** Bacry and Muzy 2003 If $2H + c\varphi(2) > 1$, then for all $t > 0$, $(Z_r^H(t))_r$ converges in L^2, when $r \to 0$.
- **Corollary** If $2H + c\varphi(2) > 1$, then $((Z_r^H(t), \quad t \in [0, T]))_r$ converges almost surely in $C([0, T], \mathbb{R})$.
- If for fixed $t > 0$ there exists $q > 2$, $\sup_r \mathbb{E}(|Z_r^H(t)|^q) < +\infty$, then $qH + c\varphi(q) \geq 1$.

Existence of MRW : Exact scaling case

- **Proposition :** If $2H + c\varphi(2) > 1$ and for $q > 2$, $qH + c\varphi(q) > 1$, then for all t, $(Z_r^H(t))_r$ converges in L^q, when $r \to 0$.

Existence of MRW : Exact scaling case

- **Proposition :** If $2H + c\varphi(2) > 1$ and for $q > 2$, $qH + c\varphi(q) > 1$, then for all t, $(Z_r^H(t))_r$ converges in L^q, when $r \to 0$.
- **Proposition :** If there exists $q \in]1, 2]$ such that $Hq + c\varphi(q) > 1$, then $t > 0$, $(Z_r^H(t))_r$ converges in L^q when $r \to 0$. Moreover, $((Z_r^H(t), \quad t \in [0, T]))_r$ converges almost surely in $C([0, T], \mathbb{R})$.

Existence of MRW : Exact scaling case

- **Proposition :** If $2H + c\varphi(2) > 1$ and for $q > 2$, $qH + c\varphi(q) > 1$, then for all t, $(Z_r^H(t))_r$ converges in L^q, when $r \to 0$.
- **Proposition :** If there exists $q \in]1, 2]$ such that $Hq + c\varphi(q) > 1$, then $t > 0$, $(Z_r^H(t))_r$ converges in L^q when $r \to 0$. Moreover, $((Z_r^H(t), \quad t \in [0, T]))_r$ converges almost surely in $C([0, T], \mathbb{R})$.
- Up to my knowledge, if $\forall q \in [1, 2], \quad qH + c\varphi(q) \leq 1$, no results of existence.

Proofs

- Since $(Z^H_{1/R}(t))_R$ is a martingale, we only have to find $q > 1$ such that $\sup_r \mathbb{E}(|Z^H_r(t)|^q) < +\infty$.

Proofs

- Since $(Z^H_{1/R}(t))_R$ is a martingale, we only have to find $q > 1$ such that $\sup_r \mathbb{E}(|Z^H_r(t)|^q) < +\infty$.
- Using independence of M and B^H,

$$\mathbb{E}(|Z^H_r(t)|^q) = C_{H,q}\mathbb{E}(|\int_{[0,T]^2} Q_r(u)Q_r(v)|u-v|^{2H-2}dudv|^{\frac{q}{2}}).$$

Proofs

- Since $(Z^H_{1/R}(t))_R$ is a martingale, we only have to find $q > 1$ such that $\sup_r \mathbb{E}(|Z^H_r(t)|^q) < +\infty$.
- Using independence of M and B^H,

$$\mathbb{E}(|Z^H_r(t)|^q) = C_{H,q}\mathbb{E}(|\int_{[0,T]^2} Q_r(u)Q_r(v)|u-v|^{2H-2}dudv|^{\frac{q}{2}}).$$

- When $q = 2$ since

$$\mathbb{E}(Q_r(u)Q_r(v)) = \exp -\varphi(2)m(C_r(0) \cap C_r(|u-v|))$$

we obtain an equivalent.

Proofs

- Since $(Z^H_{1/R}(t))_R$ is a martingale, we only have to find $q > 1$ such that $\sup_r \mathbb{E}(|Z^H_r(t)|^q) < +\infty$.
- Using independence of M and B^H,

$$\mathbb{E}(|Z^H_r(t)|^q) = C_{H,q}\mathbb{E}(|\int_{[0,T]^2} Q_r(u)Q_r(v)|u-v|^{2H-2}dudv|^{\frac{q}{2}}).$$

- When $q = 2$ since

$$\mathbb{E}(Q_r(u)Q_r(v)) = \exp -\varphi(2)m(C_r(0)\cap C_r(|u-v|))$$

we obtain an equivalent.
- When $q \neq 2$ same spirit as in Bacry and Muzy 2003.

Properties of MRW

Assume $2H + c\varphi(2) > 1$ then

- MRW Z^H has stationary independent increments.

Properties of MRW

Assume $2H + c\varphi(2) > 1$ then

- MRW Z^H has stationary independent increments.
- MRW Z^H has the following covariance structure

$$\mathbb{E}(Z_t^H Z_s^H) =$$

$$C_H \int_{[0,t]\times[0,s]} \mathbb{E}(Q_{|u-v|}(u)Q_{|u-v|}(v))|u-v|^{2H-2}dudv.$$

Properties of MRW

Assume $2H + c\varphi(2) > 1$ then

- MRW Z^H has stationary independent increments.
- MRW Z^H has the following covariance structure

$$\mathbb{E}(Z_t^H Z_s^H) =$$

$$C_H \int_{[0,t]\times[0,s]} \mathbb{E}(Q_{|u-v|}(u)Q_{|u-v|}(v))|u-v|^{2H-2}dudv.$$

- MRW Z^H has long range dependence, i.e. if

$$X_k = Z^H((k+1)\tau) - Z^H(k\tau), \quad k \in \mathbb{Z}$$

then

$$\mathbb{E}(X_k X_0) = \frac{C_H \tau^{2H}}{2H(2H-1)}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}).$$

Moments : the case of exact scaling

Proposition : Assume that for $q_0 \in [1,2]$, $Hq_0 + c\varphi(q_0) > 1$,

- $(Z^H(\lambda t))_{t\in[0,1]} =^d \lambda^H e^{\Omega_\lambda}(Z^H(t))_{t\in[0,1]}$.

Moments : the case of exact scaling

Proposition : Assume that for $q_0 \in [1,2]$, $Hq_0 + c\varphi(q_0) > 1$,

- $(Z^H(\lambda t))_{t\in[0,1]} =^d \lambda^H e^{\Omega_\lambda}(Z^H(t))_{t\in[0,1]}$.
- For $q \leq q_0$ or $q > q_0$ and $qH + c\varphi(q) > 1$,

$$\mathbb{E}(|Z^H(t)|^q) = C_q t^{qH+c\varphi(q)}.$$

Moments : the case of exact scaling

Proposition : Assume that for $q_0 \in [1,2]$, $Hq_0 + c\varphi(q_0) > 1$,

- $(Z^H(\lambda t))_{t\in[0,1]} =^d \lambda^H e^{\Omega_\lambda}(Z^H(t))_{t\in[0,1]}$.
- For $q \leq q_0$ or $q > q_0$ and $qH + c\varphi(q) > 1$,

$$\mathbb{E}(|Z^H(t)|^q) = C_q t^{qH+c\varphi(q)}.$$

- If for $q \geq 2$, $\mathbb{E}(|Z^H(t)|^q) < \infty$ then $qH + c\varphi(q) \geq 1$.

Moments : the case of exact scaling

Proposition : Assume that for $q_0 \in [1,2]$, $Hq_0 + c\varphi(q_0) > 1$,

- $(Z^H(\lambda t))_{t\in[0,1]} =^d \lambda^H e^{\Omega_\lambda}(Z^H(t))_{t\in[0,1]}$.
- For $q \leq q_0$ or $q > q_0$ and $qH + c\varphi(q) > 1$,

$$\mathbb{E}(|Z^H(t)|^q) = C_q t^{qH+c\varphi(q)}.$$

- If for $q \geq 2$, $\mathbb{E}(|Z^H(t)|^q) < \infty$ then $qH + c\varphi(q) \geq 1$.

- **Proof** : Use $(Q_{r\lambda}(\lambda t))_{t\in\mathbb{R}} =^d e^{\Omega_\lambda}(Q_r(t))_{t\in\mathbb{R}}$ with $\mathbb{E}(e^{q\Omega_\lambda}) = \lambda^{c\varphi(q)}$.

Moments : the case of scale invariance

Assume that $2H + c\varphi(2) > 1$.

- If $Z^H(t)$ has a finite p moment with $p > 2$ then $pH + c\varphi(p) > 1$.
- **Proposition** (Conjectured by Ludena 2008) If $p \in \mathbb{N}$ and $2pH + c\varphi(2p) > 1$, then $Z^H(t)$ has a finite $2p$ moment.
- **Proposition :** For $p \leq 2$ or $p \geq 2$ such that $pH + c\varphi(p) > 1$ then there exists some constants c and C such that for all $t \in (0,1]$,

$$ct^{pH+c\varphi(p)} \leq \mathbb{E}(|Z^H(t)|^q) \leq Ct^{pH+c\varphi(p)}.$$

Open problems

- What happens when $2H + c\varphi(2) < 1$ in the case of invariance scaling,
  or when $\forall q \in]1, 2], \quad qH + c\varphi(q) < 1$ in the case of exact scale invariance ?
- If $H = 1/2$, $r^{-c\varphi(2)/2}(Z_r^H(t))_{t\in[0,1]}$ converges in the sens of finite marginal distributions.
- If $H > 1/2$ we conjecture that $r^{-c\varphi(2)/2+1/2-H}(Z_r^H(t))_{t\in[0,1]}$ converges in the sens of finite marginal distributions.