Connectivity Phase Transitions in Fractal Percolation *

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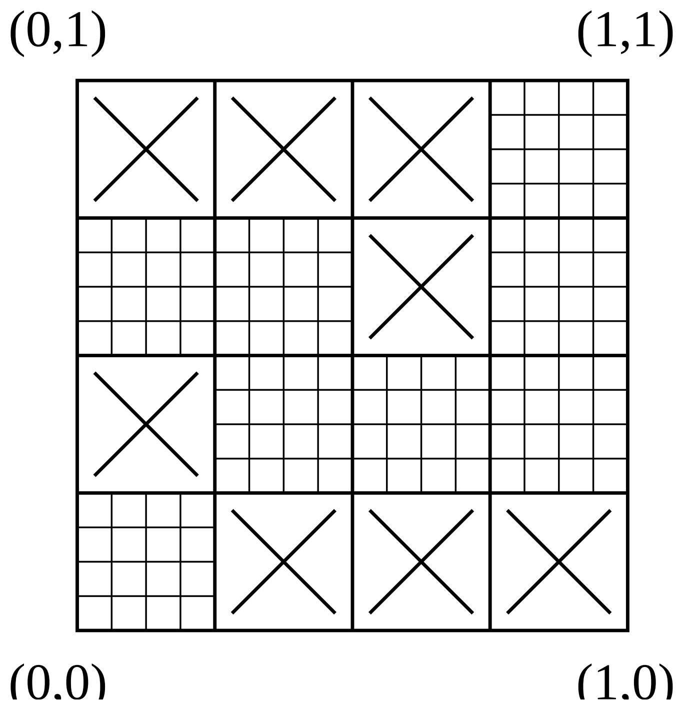
Mandelbrot's Fractal Percolation

Mandelbrot 1972, 1983

Fix $N \geq 2$ and $0 < p < 1$; let $\mathcal{C}^0 = [0, 1]^2$.



$$[0, 1]^2 \supset \mathcal{C}^1(p, N) \supset \mathcal{C}^2(p, N) \supset \ldots$$

$$\mathcal{C}(p, N) := \cap_{k=1}^{\infty} \mathcal{C}^k(p, N)$$

Some Results

Consider the event that $\mathcal{C}(p,N)$ contains a connected component crossing $[0,1]^2$ in the first coordinate direction, and let $\theta(p,N)$ denote its probability. Furthermore, define

$$p_c(N) := \inf\{p : \theta(p,N) > 0\}.$$

For all $N \geq 2$,

0. if $p \leq 1/N^2$, $\mathcal{C}(p,N)$ is almost surely empty,
1. $p_c(N) < 1$,
2. if $p < p_c(N)$, a.s. $\mathcal{C}(p,N)$ contains no connected component larger than one point,
3. $\theta(p_c(N),N) > 0$.

Three distinct phases

1. $p \leq 1/N^2$, "empty" phase

2. $1/N^2 < p < p_c(N)$, "dust" phase

3. $p \geq p_c(N)$, "connected" phase

NOTE: At $p = p_c(N)$, system is in "connected" phase.

Fractal Poisson Boolean Model

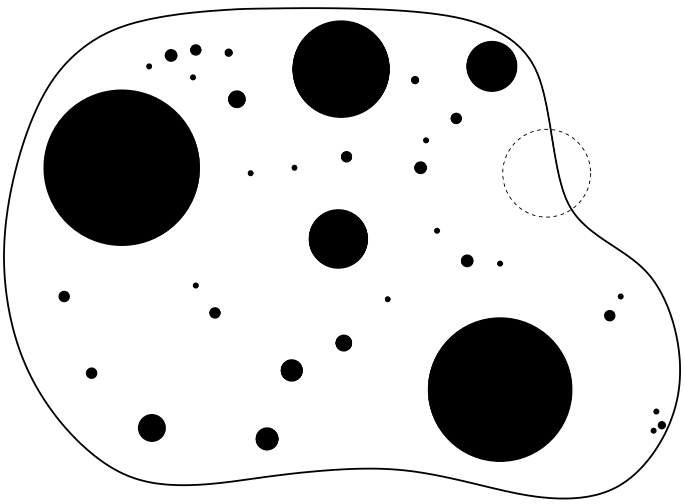
Fix $D \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$.

Let $\mathcal{P}$ be a d-dimensional Poisson point process on $D \times (0, \infty)$ with intensity $\lambda \, r^{-(d+1)} \, dx \, dr$.

Let $B(x, r) = \{y \in \mathbb{R}^d : |x - y| \leq r\}$ and define

$$\mathcal{C} := D \setminus \sum_{(x,r) \in \mathcal{P}'} B(x, r),$$

where $\mathcal{P}'$ is the thinning of $\mathcal{P}$ obtained by deleting every (x, r) such that $B(x, r) \cap D^c \neq \emptyset$.



An Alternative Definition

Let $E = E(D; a, b)$ denote the collection of d-dimensional balls of radius $r \in (a, b]$ contained in $D \subset \mathbb{R}^d$.

Let μ^{Bool} be defined by

$$\mu^{Bool}(E) = \int_a^b \int_{D_r} r^{-(d+1)} dx dr$$

where $D_r := \{x \in D : \text{dist}(x, \partial D) > r\}$.

Let P be the Poisson point process with intensity $\lambda\mu^{Bool}$. Then,

$$\mathcal{C} = D \setminus \sum_{B \in \mathsf{P}} B.$$

Let's Be More General...

Let M be the set of compact subsets K of $\mathbb{R}^d$ with nonempty interior.

Definition. We say that an infinite measure μ on M is *scale invariant* if it is invariant under the transformation $K \mapsto sK$ for all $s > 0$ (i.e., for any E such that $\mu(E) < \infty$ and any $0 < s < \infty$, $\mu(E') = \mu(E)$, where $E' = \{K : K/s \in E\}$).

Phase Transition

Let $\mathtt{P}$ be the Poisson point process with intensity $\lambda\mu$, with μ translation and scale invariant (and "locally finite"). Define

$$\mathtt{C} := D \setminus \sum_{K \in \mathtt{P}'} K,$$

where $\mathtt{P}'$ is the thinning of $\mathtt{P}$ obtained by deleting all sets that are not contained in D.

Main Theorem. There exists $\lambda_c = \lambda_c(\mu)$, with $0 < \lambda_c < \infty$, such that, with probability one, $\mathtt{C}$ contains connected components larger than one point if $\lambda \leq \lambda_c$, and is totally disconnected if $\lambda > \lambda_c$.

Idea of the Proof

Key lemma: probability of crossing an annulus is discontinuous at λ_c.

Proof of lemma based on a "renormalization"-type argument.