LAMPERTI STABLE (and associated) PROCESSES

M.E. Caballero
J.C. Pardo, J.L. Pérez
Instituto de Matemáticas,
Universidad Nacional Autónoma de México,
E-mail: marie@matem.unam.mx

Definition

An infinitely divisible probability measure μ defined on $\mathbb{R}^d$ without Gaussian component is called LAMPERTI STABLE (LS) if its Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ is given by

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$$\nu_\sigma^{\alpha,f}(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty I_B(r\xi) e^{rf(\xi)} (e^r - 1)^{-(\alpha+1)} dr, \quad B \in \mathcal{B}(\mathbb{R}^d_0), \tag{1}$$

where $\alpha \in (0,2)$, σ is non-zero finite measure on S^{d-1}, and $f : S^{d-1} \to \mathbb{R}$ is a measurable function such that $\gamma := \sup_{\xi \in S^{d-1}} f(\xi) < \alpha + 1$.

Definition

A Lévy process without gaussian component, and linear term θ, is called
LAMPERTI STABLE LÉVY PROCESS
with characteristics $L = (\alpha, f, \sigma, \theta)$ if its Lévy measure is the Lévy measure $\nu_\sigma^{\alpha,f}(dx)$ of a Lamperti stable distribution.
NOTATION: $X^L = (X_t^L, t \geq 0)$
The characteristic exponent is :
$E[\exp(i\langle y, X_t^L\rangle)] = \exp(-t\Psi(y))$ *for* $t \geq 0$, $y \in \mathbb{R}^d$

$$\Psi(y) = i\langle y, \theta\rangle + \int_{R_0^d} \Big(1 - e^{i\langle y,x\rangle} + i\langle y, x\rangle I_{\{\|x\|<1\}}\Big)\nu_\sigma^{\alpha,f}(dx)$$

In the one dimensional case since $S^0 = \{-1, 1\}$, we use the notation:
$f(1) := \beta$ and $f(-1) := \delta$.
$\sigma(\{1\}) = c_+$ and $\sigma(\{-1\}) = c_-$. and $\nu_\sigma^{\alpha,f}$ can also be written as

$$c_+ \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} I_{x>0} + c_- \frac{e^{-\delta x}}{(e^{-x} - 1)^{\alpha+1}} I_{x<0}.$$

The ideas behind this definition are : the Lamperti transformation
L_1(pssMp) $=$ Lévy process
and the α-stable Lévy process X.

Well known cases have the following characteristcs: (the constants $\sigma(\{1\}) = c_+$ and $\sigma(\{-1\}) = c_-$ are those of the original stable process $(X, \mathbf{P})$.)

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3.- If $X^\downarrow$ if the the α-stable Lévy process conditioned to hit 0 continuously, then $\xi^\downarrow = L_1(X^\downarrow)$ is LS with $\beta = \alpha\rho$ and $\delta = \alpha(1 - \rho) + 1$

and where ρ is the negativity parameter of the stable process $\mathbf{P}(X_1 < 0)$.

1.-$\nu_\sigma^{\alpha,f}$ verifies the divergence condition, i.e.

$$\int_0^\infty e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}dr=\infty \qquad \text{for any} \qquad \xi\in S^{d-1}.$$

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2.-If $\zeta<\alpha+1-\gamma$, then

$$\int_{\mathbb{R}^d} e^{\zeta\|x\|}\mu(dx)<\infty,$$

For $\kappa<\alpha+1$ and if $f\equiv\kappa$, we have

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3.-μ is moment determined.

4.-The class of infinitely divisible distributions for which the Lévy measure ν takes the following form

$$\nu(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty I_B(r\xi)\ell(\xi, r)dr, \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

is
(i) selfdecomposable if $r\ell(\xi, r)$ is decreasing for $r > 0$
or
(ii) Jurek class if $\ell(\xi, r)$ is decreasing in $r \in (0, \infty)$.

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(ii) Jurek class if $\ell(\xi,r)$ is decreasing in $r \in (0,\infty)$.
4.1.-μ belongs to the Jurek class.
4.2.-μ is selfdecomposable if $f(\xi) \leq \alpha + 1/2$, for all $\xi \in S^{d-1}$ and $\alpha \in (0,2)$.

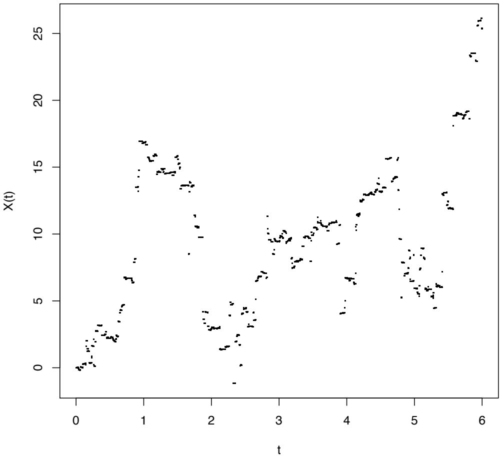
5.- For $d = 1, \mu$ has a C^{∞} density and all the derivatives of the density tend to 0 as $|x|$ tends to ∞.
6.- In the case , $d = 1$ the tail of the Lévy measure of any Lamperti stable distributions belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$, where $\beta = f(1)$ as usual.

SOME PROPERTIES for $d = 1$:

i) If $\alpha \in (1,2)$, the process X^L is a.s. of finite p- variation in every finite interval if and only if $p \in (\alpha, 2)$.

ii) The process X^L is a.s. of finite variation in every finite interval if and only if $\alpha \in (0,1)$.

- iii)
  1) If $\alpha \in (0,1)$ and $\mathbf{d} > 0$, the process X^L creeps upwards.
  2) If $\alpha \in (1,2)$ and $c_+ = 0$, the process X^L does creeps upwards.
  3) If $\alpha \in (1,2)$ and $c_+ > 0$, the process X^L does not creeps upwards.
- the point 0 is regular for $(0,\infty)$ if one of these two conditions hold:
  i) $\alpha \in [1,2)$.
  ii) $\alpha \in (0,1)$, $\mathbf{d} \geq 0$, and $c_+ > 0$.


X(t)
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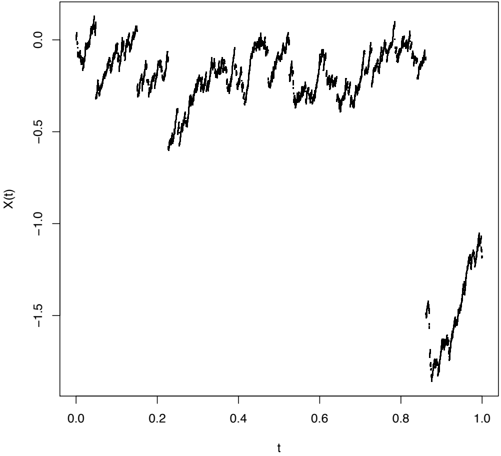
0.0
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X(t)
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t

Theorem

i)]If $\alpha \in (0,1) \cup (1,2)$, the characteristic exponent of X^L is given by

$$\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Gamma(-\alpha)\left((-i\lambda + 1 - \beta)_\alpha - (1-\beta)_\alpha\right)$$

$$-c_-\Gamma(-\alpha)\left((i\lambda + 1 - \rho)_\alpha - (1-\rho)_\alpha\right), \qquad \lambda \in \mathbb{R}.$$

ii)]If $\alpha = 1$, the characteristic exponent of X^L is given by

$$\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Big((-i\lambda+1-\beta)\psi(-i\lambda+2-\beta) - (1-\beta)\psi(2-\beta)\Big)$$

$$-c_-\Big((i\lambda + 1 - \rho)\psi(i\lambda + 2 - \rho) - (1-\rho)\psi(2-\rho)\Big), \qquad \lambda \in \mathbb{R}.$$

$\lambda \in \mathbb{R}$, $\mathcal{C}$ is the Euler constant, $\tilde{\theta}$ are constants (which are explicite)

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \qquad \text{for} \quad z \in \mathbb{C},$$

is the Pochhammer symbol and ψ is the Digamma function:

$$\psi(\omega) = \int_0^1 \frac{t^{\omega-1}-1}{\omega-1} dt - \mathcal{C}, \qquad \text{for} \quad \omega \in \mathbb{C}, \tag{2}$$

P. Patie: (case of spectrally negative and $\alpha \in (1,2)$
Recent work of Kusnetsov: introduces a widder class of Lévy measures with density:

$$\nu(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}}$$

and allows $\lambda_i \in (0,3)$
For a certaiin choise of the parametres : he obtains the tempered stable or for other parametres similar behaviour to to CGMY

$$c_+ \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} I_{x>0} + c_- \frac{e^{-\delta x}}{(e^{-x} - 1)^{\alpha+1}} I_{x<0}.$$

Rosiński (for tempered stable) and Houdré and Kawai (for layered stable processes) study the short and long time behaviour by a convinient scaling of the process. We can find similar behaviour for the Lamperti stable case.

In particular when we start with a stable process $(X, \mathbf{P}_x), x > 0$, of index α, applying the result in short time behavior after various transformations we return to this initial process. Recall that associated to the stable process three Lévy processes are obtained via the Lamperti representation of pssMp: $\xi^*, \xi^\uparrow, \xi^\downarrow$. Then the normalization of any of them according to proposition 10, converges weakly in the space of Skorokhod to the original stable process X, i.e.

$$X \xrightarrow{kill} X^* \xrightarrow{L_1} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \qquad \text{as} \quad h \to 0$$

$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{L_1} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \quad \text{as} \quad h \to 0$$

In the same spirit we could also write, using the result in long time behaviour,

$$X \xrightarrow{kill} X^* \xrightarrow{L_1} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \qquad \text{as} \quad h \to \infty,$$
$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{L_1} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \quad \text{as} \quad h \to \infty,$$

where W is a centered brownian motion.

we find two examples related to the factorization (BY)

$$\mathbf{e} \stackrel{law}{=} \mathbf{e}^{\alpha} \tau_{\alpha}^{-\alpha}.$$

where $\mathbf{e}$ is an exponential variable independent of the α-stable variable τ_{α}. The first of them is related with the exponential functional of a killed subordinator Z^1 whose Laplace exponent is given by

$$\phi_1(\lambda) = \frac{\Gamma(\alpha\lambda + 1)}{\Gamma(\alpha(\lambda - 1) + 1)}.$$

It is easy to see that it is related to the Laplace exponent Φ_L of a Lamperti stable subordinator X^L with characteristics $(\alpha, \alpha, \sigma, \theta)$, zero drift, and $\sigma(\{1\}) = \alpha/\Gamma(1-\alpha)$. The relationship between both Laplace exponents is

$$\phi_1(\lambda) = \Phi_L(\alpha\lambda) + \frac{1}{\Gamma(1-\alpha)}.$$

This is also studied by Rivero. He finds its renewal density and

The Laplace exponent of the second subordinator, here denoted by Z^2, is given by

$$\phi_2(\lambda) = \lambda \frac{\Gamma(\alpha(\lambda - 1) + 1)}{\Gamma(\alpha\lambda + 1)},$$

and can be expressed in terms of the Laplace exponent $\Phi_{L,2}$ of a Lamperti stable subordinator $X^{L,2}$ with characteristics $(1 - \alpha, 1, \sigma, \theta)$, and zero drift where $\sigma(\{1\}) = \alpha/\Gamma(1 - \alpha)$. The relation between them is

$$\phi_2(\lambda) = \alpha \Phi_{L,2}(\alpha\lambda).$$

In both cases this allows us to compute the law of the exponential functional of αX^L and $\alpha X^{L,2}$ in terms of the one of Z^1 and Z^2, respectively.

There is another example in (BY) which is related to the factorization

$$\mathbf{e} \stackrel{law}{=} \gamma_s^{\alpha} J_s^{(\gamma)},$$

where $s \geq \alpha$, γ_s is a Gamma r.v. with parameter s and $J_s^{(\gamma)}$ denotes a certain r.v which is independent of γ_s. In this case, the killed subordinator related to the exponential functional which has the same moments as the γ_s, can be expressed as the sum of two independent Lamperti stable procesess. In further calculation are carried over concerning this subordinator.

In the paper (Salminen, Vallois, Yor) in section 5.3, the authors found the Lévy measure of the inverse of the local time at 0 of an Ornstein Uhlenbeck process driven by a standard Brownian motion and parameter $\gamma > 0$. This measure is

$$\nu(t) = \frac{\gamma^{3/2} e^{\gamma t/2}}{\sqrt{2\pi}(\sinh(\gamma t))^{3/2}} = \frac{(2\gamma)^{3/2} e^{2\gamma t}}{\sqrt{2\pi}(e^{2\gamma t} - 1)^{3/2}}$$

and the corresponding Laplace exponent is computed. It is related to a Lamperti stable distribution with characteristics $(1/2, 1, \sqrt{\gamma/\pi})$.

In the work of (KPRi) where they study N-tuple laws of LP, we find in the examples the process $\xi^{\uparrow}$, $L = (\beta = \alpha\rho + 1, \delta = \alpha(1-\rho), 0)$, and the authors find a Wiener Hopf factorization for it:
At the same time in another example they use two LS subordinators which play the role of the ascending (descending) ladder height process of a Lévy process X. They also find a Wiener Hopf factorization for X.

In a paper by (KR) they also find relations between some LS processes and the parent process. Subordinator $\{X^L, L = (\alpha, \beta, \sigma, \theta)\}$ with zero drift and a special killing rate K. Subordinator, Y, with no drift and no killing rate. The Laplace exponent of Y satisfies

$$\phi_Y(\lambda) = \frac{\lambda}{\phi_{X_L}(\lambda)}, \qquad \text{for} \quad \lambda \geq 0,$$

Its parent process Y^P has Laplace exponent

$$\psi_{Y^P}(\lambda) = \frac{\lambda^2 \Gamma(1 - \beta + \lambda)}{\Gamma(1 - \beta + \lambda + \alpha)}.$$

its associated scale function is given by

$$W_{Y^P}(x) = -Kx + c_+ \sum_{n\geq 0} \frac{(\alpha+1)_n(\alpha-\beta)_n}{n!(\alpha+2-\beta)_n}\Big(1-e^{-(\alpha+2-\beta+n)x}\Big), \quad x \geq 0.$$

Now let $Y^{*,P}$, be the parent process of the Lamperti subordinator X^L with killing rate K, is a spectrally negative Levy process which drifts to ∞ and whose Laplace exponent is given

$$\psi_{Y^{P,*}}(\lambda) = \frac{c_+\Gamma(-\alpha)\lambda\Gamma(\lambda+1-\beta+\alpha)}{\Gamma(\lambda+1-\beta)}.$$

It is important to note that the processes Y^P and $Y^{P,*}$ are the sum of two independent Lamperti stable processes and that they have been recently used for the risk neutral stock price model by Eberlein and Madan.

ANOTHER EXAMPLE OF THE LAMPERTI TRANSFORMATION RELATED TO LAMPERTI STABLE PROCESSES:

$Z = (Z_t, t \geq 0)$ is a symmetric stable Lévy process of index $\alpha \in (0, 2)$ in R^d $(d \geq 1)$. The characteristic exponent is:

$$\mathbb{E}_0 \left(\exp\{i < \lambda, Z_t >\} \right) = \exp\{-t|\lambda|^{\alpha}\},$$

for all $t \geq 0$ and $\lambda \in R^d$. Here $\mathbb{P}_z$ denotes the law of the process Z initiated from $z \in^d$ and $< \cdot, \cdot >$ the Euclidean inner product.

Since Z is symmetric and satisfies the scaling property with index α, i.e. for every $b > 0$

$$\textit{The law of } (bZ_{b^{-\alpha}t},\, t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{bx}, \tag{3}$$

the radial proces $R_t = |Z_t|$ is Markov process satisfying the scaling property with the same index α. So it will be a pssMp and we can obtain the generator and from this generator its Lamperti Transformation and the corresponding Lévy process with its triplet.

Proposition

If $f : [0,\infty) \to$ is such that f and xf' are continuous on $[0,\infty)$. The infinitesimal generator M of $R = (R_t, t \geq 0)$, acts as follows for $a > 0$,

$$Mf(a) = \frac{2^{\alpha+1}(d/2)_\alpha}{a^\alpha} \int_0^\infty \left(f(\rho a) - f(a) - l(\log \rho) f'(a)\right) dH(\rho)$$

where

$$dH(\rho) = \frac{\rho^{d-1} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right)}{(1+\rho^2)^{\alpha+d/2}} d\rho,$$

$$\bar{F}(z) = \mathcal{F}_{2,1}((\alpha+d)/4, (\alpha+d)/4+1/2; d/2; z), z \in (-1,1),$$

$\mathcal{F}_{2,1}$ is the Gauss's hypergeometric function.
and

$l(y) = \frac{y}{1+y^2} e^{(1-d)y}(1+e^{2y})^{\alpha+d/2-1} 1_{A_\epsilon}(e^y)$. The function l i satisfies the following:

- Is a bounded Borel function.
- And also that $l(y) \sim y$ as $y \to 0$.

Idea of the proof:

$$Z_t = B_{2\sigma_t}$$

with B a d-dimensional Brownian motion initiated from $x \in R^d$ and let $\sigma = (\sigma_t, t \geq 0)$ is an independent stable subordinator with index $\alpha/2$ initiated from 0 so

$$R = |B|_{2\sigma_t}$$

Finally to obtain the Lévy measure of $\xi = L_1(R)$

$$\Pi(du) = H(du) \circ e^u,$$

$$\begin{aligned}\Pi(du) &= e^u \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{(d-1)u}}{(1+e^{2u})^{(\alpha+d)/2}} \bar{F}\left(\frac{4e^{2u}}{(e^{2u}+1)^2}\right) du \\ &= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{du}}{(1+e^{2u})^{(\alpha+d)/2}} \bar{F}\left(\frac{4e^{2u}}{(e^{2u}+1)^2}\right) du. \quad (4)\end{aligned}$$

In dimension one:

Proposition

Let us consider that $d = 1$, then we have the following decomposition for the process ξ:

$$\xi \overset{\mathcal{L}}{=} \xi^1 + \xi^2 \tag{5}$$

where $\xi^1 = (\xi^1_t, t \geq 0)$ is a Lamperti stable Lévy process, and $\xi^2 = (\xi^2_t, t \geq 0)$ is a compound Poisson process independent of ξ^1.

Π_1 and Π_2 are given respectively by

$$\Pi_1(dy) = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)}\left(\frac{e^y}{(e^y-1)^{\alpha+1}}1_{\{y\geq 0\}} + \frac{e^y}{(1-e^y)^{\alpha+1}}1_{\{y<0\}}\right) dy$$

$$\Pi_2(dy) = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)}\frac{e^y}{(e^y+1)^{\alpha+1}}dy.$$

Theorem

Let $Z = (Z_t, t \geq 0)$ *be a symmetric* α*-stable process and consider the Lamperti transform* ξ *of* $R = \|Z\|$*. Then there exists a constant* k' *related to the normalization of the local time such that the characteristic exponent of the process* ξ *is given by*

$$\psi(\lambda) = k'\alpha 2^{\alpha-1} \frac{\Gamma(d/2)\Gamma(-\alpha/2)}{\Gamma(1-\alpha/2)\Gamma(\alpha/2)} \left(\frac{i\lambda + d}{2}\right)_{\alpha/2} \left(-\frac{i\lambda}{2}\right)_{\alpha/2}.$$

We have obtained the characteristic exponent for the process $\xi^{\alpha,d}$ in the case where $\alpha < d$ using the Wiener Hopf factorization. We will now see that the same formula holds true in the case $\alpha = d = 1$, case which has been studied in another context by CPY [*]. They consider the absolute value of a Cauchy process $X = |C|$ which is our pssMp in the case $\alpha = d = 1$. They find the characteristic exponent of the associated process via the Lamperti transfomation which we call $\xi^{(1,1)}$:

$$E[\exp i\lambda\xi_t^{(1,1)}] = \exp(-\lambda \tanh(\frac{\pi\lambda}{2})), \quad t \geq 0, \quad \lambda \in R$$

as well as other properties of $\xi^{(1,1)}$.

$$\left(\frac{i\lambda+1}{2}\right)_{1/2}\left(-\frac{i\lambda}{2}\right)_{1/2} = \frac{|\Gamma\left(\frac{i\lambda+1}{2}\right)|^2}{|\Gamma\left(\frac{i\lambda}{2}\right)|^2} = \frac{\frac{\pi}{\cosh(\pi\lambda/2)}}{\frac{\pi}{(\lambda/2)\sinh(\pi\lambda/2)}}$$

As a consequence,

$$\psi^{1,1}(\lambda) = k\alpha 2^{\alpha-1}\frac{\Gamma(-1/2)}{\Gamma(1/2)}\frac{\frac{\pi}{\cosh(\pi\lambda/2)}}{\frac{\pi}{(\lambda/2)\sinh(\pi\lambda/2)}}$$