Fractal percolation

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Outline

Background (the model and preliminary results)

Ordinary percolation

The main question

The answer and a Corollary

The model

We consider first the case $d = 2$. Let $0 < p < 1$ and N be an integer. Partition the unit square into N^2 subsquares in the canonical way.

For every square, retain (or discard) it independently with probability p (or $1 - p$). This will yield a random set $\mathcal{C}^1(p, N) \subset [0,1]^2$.

For every retained subsquare repeat the procedure on a smaller scale thus obtaining $\mathcal{C}^2(p, N) \subset \mathcal{C}^1(p, N) \subset [0,1]^2$. Continue the procedure on all scales.

Define

$$\mathcal{C} := \cap_{k=1}^{\infty} \mathcal{C}^k(p, N).$$

Observe the scaling invariance!

Some known results ($d = 2$)

Consider the event that $\mathcal{C}$ contains a connected component which intersects both $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$.

Let $\theta(p, N)$ denote the probability of this event. An easy coupling shows that $\theta(p, N)$ is non-decreasing in p. It is therefore natural to define

$$p_c(N) := \inf\{p : \theta(p, N) > 0\}.$$

It is known that $0 < p_c(N) < 1$ and that

$$\theta(p_c(N), N) > 0.$$

Ordinary percolation

Consider the lattice $\mathbb{Z}^2$. We let $\eta \in \{0,1\}^{\mathbb{Z}^2}$ be such that $\eta(x) = 1$ with probability p and $\eta(x) = 0$ otherwise, and this is done independently for every $x \in \mathbb{Z}^2$. If $\eta(x) = 1$ we say that x is open and otherwise it is closed.

Let $\mathcal{A}$ denote the set of $x \in \mathbb{Z}^2$ that are connected to the origin via a path of open sites. It is well known that there exists a $0 < p_c < 1$ such that

$$p_c = \inf\{p : \mathbb{P}_p(|\mathcal{A}| = \infty) > 0\}.$$

Ordinary percolation

However, instead of considering this model on the full lattice it is sometimes useful to consider only the part $\{0, \ldots, N\}^2$. In this setting one often considers the event that there exists an open path intersecting both $\{0\} \times \{0, \ldots, N\}$ and $\{N\} \times \{0, \ldots, N\}$. If we denote this event by $\mathcal{C}_N$ it is known that

$$\lim_{N\to\infty} \mathbb{P}_p(\mathcal{C}_N) = \begin{cases} 1 & \text{if } p > p_c \\ 0 & \text{if } p < p_c. \end{cases}$$

The question what happens *at* p_c has attracted major attention for many years.

Interesting question

In fact, instead of letting $N \to \infty$ one considers the so called *scaling limit*. Here, one considers only the unit box, but let the lattice spacing go to 0.

In particular one would believe that for sitepercolation on $\mathbb{Z}^2$,

$$\lim_{N \to \infty} \mathbb{P}_{p_c}(\mathcal{C}_N) \in (0,1).$$

When taking the scaling limit in ordinary percolation, one removes the intrinsic scale of the system. What is the analogue of this in the fractal percolation model? The size of the largest scale is N^{-1}. Therefore, we wanted to investigate what happens with $\theta(p_c(N), N)$ when $N \to \infty$. This question is valid for any $d \geq 2$.

Preliminaries of the proof

It is also known that $p_c(N) \to p_c^{site}$ where p_c^{site} is the critical density for ordinary site percolation on the lattice $\mathbb{L}^d$. This lattice has the same set of sites as $\mathbb{Z}^d$ but x, y are joined by an edge iff $|x_i - y_i| \leq 1$ for every i with equality for at most $d-1$ of the i.

In fact Falconer and Grimmett proved that for any $p > p_c^{site}$ we have that $\lim_N \theta(p, N) = 1$.

Interesting question

One might believe that the result that $\lim_N \theta(p, N) = 1$ for any $p > p_c^{site}$ would give us some information.

As pointed out above, in the case of ordinary percolation it is well known that the probability of having an open crossing in an $N \times N$ box also tends to 1 when $p > p_c$. However at the critical point the crossing probability converges to something strictly between 0 and 1.

Hence apriori we do not know what to expect. Any guesses?

Answer and consequences

It turns out that the answer (for any $d \geq 2$) is that

$$\lim_{N \to \infty} \theta(p_c(N), N) = 1.$$

As a biproduct we get that for any $d \geq 2$ there exists an N_1 such that for every $N \geq N_1$ we have that

$$\theta(p_c(N), N) > 0,$$

i.e. we have a discontinuity!