Divergence Free Elliptic Gaussian Vector Random Fields

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Plan

Gaussian processes

General definitions

Definition

Let $(X(t) = (X_1(t), ..., X_d(t)), t \in \mathbb{R}^d)$ a vectorial centered random field.

- . Let $0 < H < 1$, X is H Self-Similar if

$$(X(\lambda t), t \in \mathbb{R}^d) =_{Law} (\lambda^H X(t), t \in \mathbb{R}^d), \quad \forall \lambda > 0.$$

- X is with stationnary increments if

$$(X(t+h) - X(h), t \in \mathbb{R}^d) =_{Law} (X(t), t \in \mathbb{R}^d), \forall h \in \mathbb{R}^d.$$

- X is elliptic of degree $d + 2H$ if there exist $0 < c \leq C < \infty$ such that,

$$c|x|^{2H} \leq \mathbb{E}|X(x)|^2 \leq C|x|^{2H}, \forall x \in \mathbb{R}^d.$$

Divergence free vector random fields

Problem

Definition

A d dimensional vector random field is weakly divergence free, if

$$< X, \nabla\varphi >= 0 \ a.s.\forall \text{ regular function } \varphi \text{ on } \ \mathbb{R}^d.$$

Question

Is it possible for a vector random field to enjoy self-similarity, stationnarity of the increments, ellipticity properties, and furthermore to be weakly divergence free ?

The gaussian situation

Let X be a centered gaussian random field.

Reproducing Kernel Hilbert Space (RKHS)

The covariance $R(t,s) = \mathbb{E}(X(t)X(s)); t,s \in \mathbb{R}^d$ generate a RKHS H_X

$$H_X = sp\{(R(t,.)t \in \mathbb{R}^d\},$$

equiped with the scalar product $<,>_{H_X}$ such that

$$< R(t,.), R(.s) >_{H_X} = R(t,s).$$

The gaussian situation

Gaussian random fields and orthonormal basis of H_X

Let $(\Phi_n, n \in \mathbb{N})$ an orthonormal basis of H_X,

$$< \Phi_n, \Phi_m >_{H_X} = \delta_{nm}.$$

Then the centered gaussian process X can be expressed the following way,

$$X(t) = \Sigma_n \Phi_n(t) \theta_n,$$

for an iid sequence of normal gaussian random variables (θ_n).
The series is almost surely converging in $L^2(\mathbb{R}^d)$.

Orthonormal divergence free wavelets

Divergence free vector space

Let us consider the Hilbert space

$$L^2_{FD}(\mathbb{R}^d) = \{h \in (L^2(\mathbb{R}^d)^d | div(h) \in L^2(\mathbb{R}^d), div(h) = 0\}.$$

Let $u_\xi = \frac{\xi}{|\xi|}$ and Π_ξ the hyperplane orthogonal to ξ, with $d(O, \Pi_\xi) = 1$. Let's give a measurable function $\xi \to (\omega^1(\xi), ..., \omega^{d-1}(\xi), u_\xi)$ to the orthonormal bases of $\mathbb{R}^d$.

Ordinary wavelets Meyer's wavelets

Let $(\psi^\epsilon, \epsilon \in \{0,1\}^d - (0, ..., 0))$ a family of basic wavelets of $L^2(\mathbb{R}^d)$ such that we get the orthonormal wavelets basis

$$(\psi_\lambda(x) := 2^{j\frac{d}{2}} \psi^\epsilon(2^j x - k), \lambda = (\epsilon, j, k); \ \lambda \in \Lambda).$$

Divergence free wavelet basis

Construction of a DF wavelets basis

Let us define the family of vector functions $(\Psi_\lambda^b, b = 1, .., d-1, \lambda \in \Lambda)$ by setting

$$\Psi_\lambda^b(x) = \int_{\mathbb{R}^d} e^{i<x,\xi>} \omega^b(\xi) \hat{\psi}_\lambda(\xi) d\xi.$$

Lemma

The family $(\Psi_\lambda^b, b = 1, .., d-1, \lambda \in \Lambda))$ *is an orthonormal basis of* $L^2_{FD}(\mathbb{R}^d)$.

Proof

$$1) \;\; < \Psi_\lambda^b, \Psi_{\lambda'}^{b'} >= \int < \omega^b(\xi), \omega^{b'}(\xi) > \hat{\psi}_\lambda(\xi) \overline{\hat{\psi}_{\lambda'}(\xi)} d\xi = \delta_{bb'} \delta_{\lambda\lambda'}.$$

$$2) \;\; < h, \Psi_\lambda^b >= \int < \hat{h}, \omega^b > \hat{\psi}_\lambda d\xi = 0 \;\; \forall (b, \lambda) \Rightarrow \hat{h} = 0, \text{ so } h = 0.$$

About divergence free wavelets

Comment

- Even if ψ_ϵ where with compact support, Ψ_ϵ^b cant be such.
- A result by P.G. Lemarié state that an orthonormal divergence free wavelet basis with compact support cannot exists.
- If we want divergence free wavelets with compact support, the only possibility is to consider bi-orthogonal divergence free wavelets bases cf (Deriaz, Perrier) for example.

Gaussian white noise on $L^2_{DF}(\mathbb{R}^d)$

Construction of the white noise on $L^2_{DF}(\mathbb{R}^d)$

Let $(\theta^b_\lambda, b = 1, ..., d-1, \lambda \in \Lambda)$ an iid family of gaussian normal random variables.

Lemma

The vectorial random distribution

$$W_{DF}(x) = \Sigma_{b=1}^{d-1} \Sigma_{\lambda \in \Lambda} \Psi^b_\lambda(x) \theta^b_\lambda,$$

is the gaussian white noise on $L^2_{DF}(\mathbb{R}^d)$.

Proof

Let $h, k \in L^2_{DF}(\mathbb{R}^d)$. We have

$$h(resp.k) = \Sigma_{b=1}^{d-1} \Sigma_{\lambda, \in \Lambda} h_\lambda^b (resp.k_\lambda^b) \Psi_\lambda^b.$$

So

$$\mathbb{E}(< W_{DF}, h >< W_{DF}, k >) =$$

$$\Sigma_{b,b'=1}^{d-1} \Sigma_{\lambda,\lambda' \in \Lambda} \int < \hat{h}(\xi), \omega^b > \overline{< \hat{k}, \omega^b >} \psi_\lambda(\xi) \overline{\psi_{\lambda'}(\xi)} d\xi =$$

$$\Sigma_{b,b=1}^{d-1} \Sigma_{\lambda \in \Lambda} h_\lambda^b k_\lambda^b =< h, k >_{L^2_{DF}},$$

thanks to lemma.

DF Elliptic pseudodifferential operator

Symbol

Let $\xi \to S(u_\xi), S^{-1}(u_\xi)$ two measurable functions such that for $0 < c \leq C < \infty$

$$c \leq \ < S(u_\xi)(resp.S^{-1}(u_\xi))\omega^b(\xi), \omega^b(\xi) > \ \leq C, b = 1, .., d-1,$$

$$S(u_\xi)(resp.S^{-1}(u_\xi))u_\xi = 0,\ S(u_\xi)S^{-1}(u_\xi)\omega^b(\xi) = \omega^b(\xi), b = 1, .., d-1.$$

For $0 < H < 1$ and 'DF' matrix functions S, let us define the symbol $a(\xi)$ and the operator A acting on $L^2_{DF}(\mathbb{R}^d)$ by setting

$$a(\xi) = |\xi|^{\frac{d}{2}+H} S(u_\xi), \quad Ah = \int e^{i<x,\xi>} a(\xi)\hat{h}(\xi)d\xi.$$

Let us define the Hilbert space H_A,

$$H_A = \{h \in L^2_{DF}, h(0) = 0 \text{ with scalar product } < h, k >_A = < Ah, Ak >_{DF}\}.$$

RKHS

Proposition

Let the symbols S, S^{-1} be given and satisfying the former properties. For any $f \in L^2_{DF}$, there exist a unique $h \in H_A$ such that

$$h(0) = 0; \ Ah = f \quad \text{and} \quad A^{-1} f = \int \frac{(e^{i<x,\xi>} - 1)}{|\xi|^{\frac{d}{2}+H}} S^{-1}(u_\xi) \hat{f}(\xi) d\xi.$$

RKHS

Let us define the set of functions $(\Gamma_\lambda^b, b = 1, .., d-1, \lambda \in \Lambda)$ by setting

$$\gamma_\lambda^b = A^{-1}\Psi_\lambda^b,$$

$$\Gamma_\epsilon^b = A^{-1}\Psi_\epsilon^b, \; \Gamma_\lambda^b(x) = \Gamma_\epsilon^b(2^j x - k), \lambda = (\epsilon, j, k).$$

Basis

Proposition

The family $(\gamma_\lambda^b, b = 1, .., d-1; \lambda \in \Lambda)$ *is an orthonormal basis of the Hilbert space* H_A*. Furthermore*

$$\gamma_\lambda^b = 2^{-jH}\Gamma_\lambda^b, b = 1, .., d-1; \lambda = (\epsilon, j, k).$$

Divergence free self similar vector gaussian processes with stationary increments

Let S, S^{-1} the above matrix functions and given $0 < H < 1$, then let's A the associated pseudo differential operator.

representations of X

Theorem

With the above elements, let X a centered vector gaussian process with RKHS H_A. Then, if W_{DF} is the white noise on $L^2_{DF}(\mathbb{R}^d)$, X is expressed by

$$X(x) = \int \frac{(e^{i<x,\xi>} - 1)}{|\xi|^{\frac{d}{2}+H}} S^{-1}(u_\xi) \hat{W}_{DF}(d\xi).$$

Divergence free self similar vector gaussian processes with stationary increments

Series decomposition

Theorem

Under the former hypotheses, the vector gaussian process X admits the divergence free adapted wavelets series decomposition

$$X(x) = \Sigma_{b=1}^{d-1} \Sigma_{\lambda \in \Lambda} 2^{-jH} \Gamma_{\lambda}^{b}(x) \theta_{\lambda}^{b},$$

for a iid sequence $(\theta_{\lambda}^{b}, b = 1, .., d-1, \lambda \in \Lambda)$ of gaussian normal random variables.
The series almost surely, uniformly comverges on every compact.
X is weakly divergence free, H self similar, with stationnary increments.

Simulating divergence free self similar vector gaussian processes

Problems

- The (γ_λ^b)'s are with $\mathbb{R}^d$ as support.
- There is no reason to find orthonormal bases of H_A with compact support.
- The fact that the (θ_λ^b)'s are iid is very interesting. This gives hope to obtain a fast simulation algorithm of divergence free gaussian vectors fields.
- What is to be relaxed in order to solve our simulation problem ?

Simulating divergence free self similar vector gaussian processes

Solution

- Mimicking the Lévy decomposition of the brownian motion.
- So we must find an alternative to the γ_λ^b's, with compact support such that $supp\gamma_\lambda^b = [-1,1]^d$.
- This exlclude the Deriaz Perrier divergence free basis.
- We shall begin in considering the Sutter proposition.

Sutter proposition

Let $D = (D_{ij}; i, j = 1, .., d)$ the following matrix operator

$$D_{ii} = \Delta - \partial_i^2; \; D_{ij} = -\partial_i \partial_j.$$

Let φ such that $supp\varphi = [-1, 1]^d$. Let us set

$$(\phi_{ij}(x)) = D\varphi(x).$$

Lemma

$$\forall a \in \mathbb{R}^d, div(\phi(x)a) = 0.$$

The challenge is, from the matrix function ϕ, to built another matrix function $\Phi(x)$ such that, if $\Phi_\lambda(x) = \Phi(2^j x - k)$, then

$$Closure_{L^2_{DF}}[sp\{\Phi_\lambda(.)a_\lambda, \lambda \in \Lambda, a_\lambda \in \mathbb{R}^d\}] = L^2_{DF}.$$

Such a Φ is said with completion property.

Simulating divergence free self similar vector gaussian processes

For $0 < H < 1$, given a iid family $(\theta_\lambda, \lambda \in \Lambda)$ of centered normal gaussian vectors with covariance I_d, let us define the divergence free gaussian vector process Y by setting

$$Y(x) = \Sigma_{\lambda \in \Lambda} 2^{-jH} \Phi_\lambda(x) \theta_\lambda.$$

The next result is a partial justification of the proposition of simulation.

Proposition

If for any $z \in \mathbb{R}^d$, in the limiting set of laws

$$\{Lim_{r \to 0} \frac{Y(z+rx) - Y(z)}{r^H}\},$$

there exists a constant 'fiber', then Φ enjoy the completion property.

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