Monochromatic arm exponents for critical 2D percolation

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Outline

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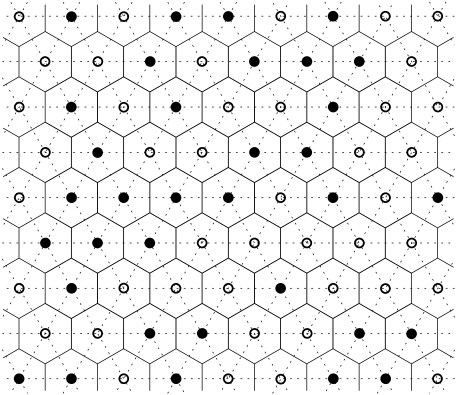
1. Critical percolation in 2D
   - Standard percolation background
   - Critical percolation
   - Arm exponents
2. Existence of the monochromatic exponents
3. Comparison with the polychromatic exponents
   - Heuristics
   - A correlation inequality
   - The strict inequality

Site percolation

We consider site percolation on the triangular lattice:



Site percolation – definition

We color the sites randomly: For a parameter $p \in [0, 1]$,

- Each site is *black* (occupied) with probability p, *white* (vacant) with probability $1 - p$.
- The sites are *independent* of each other.

Site percolation – phase transition at $p_c = 1/2$

There is a *phase transition* at $p_c = 1/2$:

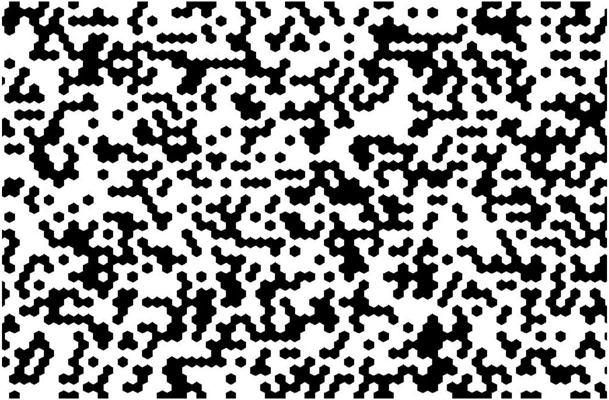
- If $p < 1/2$: a.s. no infinite cluster (*sub-critical* regime).
- If $p > 1/2$: a.s. a *unique* infinite cluster (*super-critical* regime).

If $p = 1/2$: *critical* regime, a.s. no infinite cluster.

Exponential decay – $p < p_c$

In the sub-critical regime ($p < 1/2$),

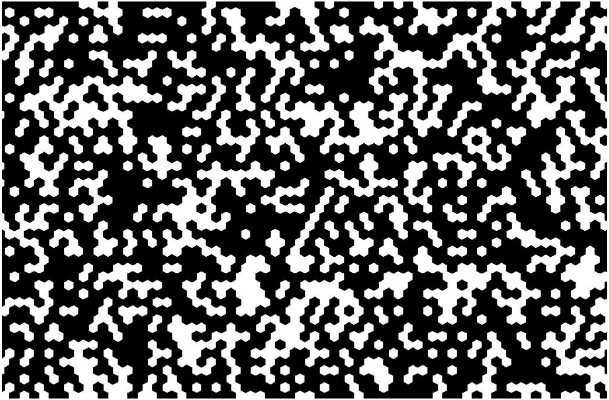
$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n) \leqslant e^{-C(p)n}.$$



Exponential decay – $p > p_c$

In the super-critical regime ($p > 1/2$),

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n | 0 \not\rightarrow \infty) \leqslant e^{-C(p)n}.$$



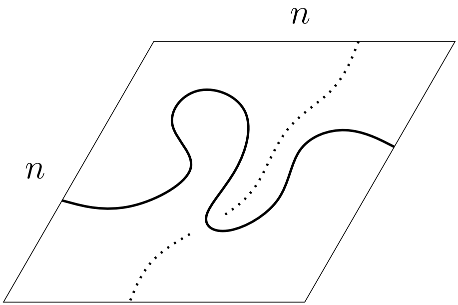
Critical regime

At the critical point ($p = p_c$), there is no characteristic length:

Crossing probabilities – *a priori* estimate

For symmetry reasons, we have for example (we denote by $\mathcal{C}_H$ the existence of a left-right crossing):

$$\mathbb{P}_{1/2}(\mathcal{C}_H([0,n]\times[0,n])) = 1/2.$$



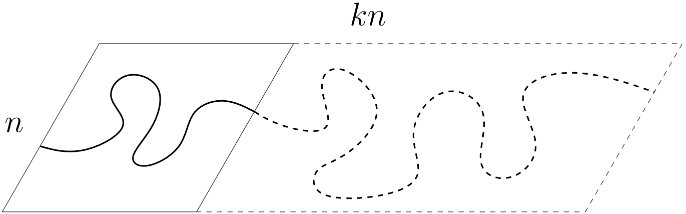
Crossing probabilities – general case

This implies the Russo-Seymour-Welsh theorem (key tool):

Theorem (Russo-Seymour-Welsh)

For each $k \geqslant 1$, there exists $\delta_k > 0$ such that

$$\mathbb{P}_{1/2}(\mathcal{C}_H([0, kn] \times [0, n])) \geqslant \delta_k.$$

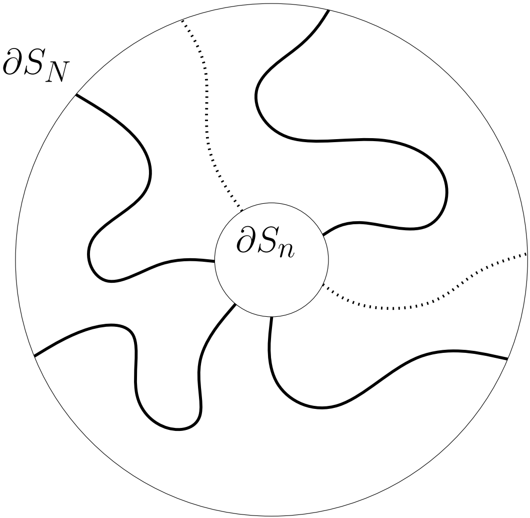


Conformal invariance

A precise description of critical percolation was made possible by

- the introduction of SLE processes by Schramm, and its subsequent study by Lawler, Schramm and Werner;
- the proof of conformal invariance of critical percolation in the scaling limit by Smirnov.

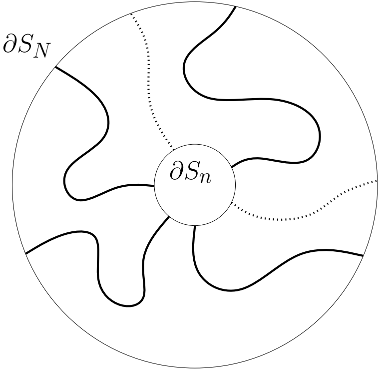
We will consider *arm events* like this:



Arm events – notation

We denote them by $\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N$:

- j = number of arms (e.g. $j = 6$),
- σ = sequence of colors (e.g. $\sigma = WBWBBB$).



Arm events – two key facts

- An *a priori* bound:

$$\left(\frac{n}{N}\right)^{\varepsilon} \geqslant \mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N) \geqslant \left(\frac{n}{N}\right)^{1/\eta_j}.$$

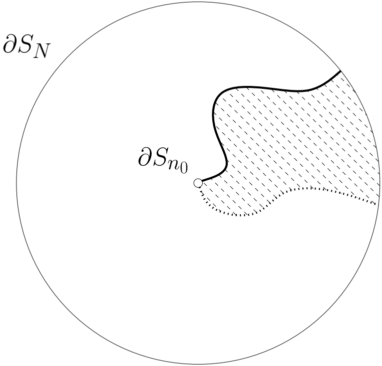
- Quasi-multiplicativity:

$$\begin{aligned} C_1 \mathbb{P}_{1/2}(\partial S_{n_1} \rightsquigarrow_{j,\sigma} \partial S_{n_2}) \mathbb{P}_{1/2}(\partial S_{n_2} \rightsquigarrow_{j,\sigma} \partial S_{n_3}) \\ \leqslant \mathbb{P}_{1/2}(\partial S_{n_1} \rightsquigarrow_{j,\sigma} \partial S_{n_3}) \\ \leqslant C_2 \mathbb{P}_{1/2}(\partial S_{n_1} \rightsquigarrow_{j,\sigma} \partial S_{n_2}) \mathbb{P}_{1/2}(\partial S_{n_2} \rightsquigarrow_{j,\sigma} \partial S_{n_3}) \end{aligned}$$

The "color-exchange trick"

For any two σ, σ' both non-constant:

$$\mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,\sigma} \partial S_n) \asymp \mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,\sigma'} \partial S_n)$$



Polychromatic arm exponents

Theorem (Lawler, Schramm, Werner, Smirnov)

For every $j \geqslant 2$, and for every non-constant sequence σ,

$$\mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,\sigma} \partial S_n) = n^{-\alpha_j + o(1)},$$

where the exponent is known exactly:

$$\alpha_j = \frac{j^2 - 1}{12}.$$

The α_j are known as the *polychromatic arm exponents*.

Monochromatic arm exponents – $j = 1$

Theorem (Lawler, Schramm, Werner, Smirnov)

We have

$$\mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_n) = n^{-\alpha_1' + o(1)},$$

where α_1' is known exactly:

$$\alpha_1' = \frac{5}{48}.$$

This exponent α_1' does not fit in the previous family (α_j). Formally it is equal to $\alpha_{3/2}$ ...

Monochromatic arm exponents – general case

What happens to the arm probability

$$\mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,\sigma} \partial S_n)$$

for $\sigma = BB \dots B$ constant (monochromatic case)?

- Does it still follow a power law when $j \geqslant 2$?
- Are the exponents the same as in the polychromatic case?
- If not, how do they compare?

Main results – existence of the exponents

Theorem (B., Nolin)

For each $j \geqslant 2$, for $\sigma = BB \dots B$ constant,

$$\mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,\sigma} \partial S_n) = n^{-\alpha'_j + o(1)},$$

for some exponent α'_j.

The values of the α'_j (for $j > 1$) are *not* known exactly. α'_2 is physically relevant: known as the *backbone exponent*.

Main results – monochromatic *vs.* polychromatic

Theorem (B., Nolin)

For every $j \geqslant 2$, the following inequalities hold:

$$\alpha_j < \alpha'_j < \alpha_{j+1}.$$

In particular, the (α'_j) form a different family of exponents.

Remark: in the half-plane case, the color-exchange trick implies that mono- and poly-chromatic are the same.

First proof: $\alpha'_j < \alpha_{j+1}$

The inequality is a direct consequence of the *FKG inequality*:

$$\begin{aligned}
&\mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j+1,B\ldots BW} \partial S_n) \\
&\qquad = \mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,B\ldots B} \partial S_n \;\cap\; \partial S_{n_0} \rightsquigarrow_{1,W} \partial S_n) \\
&\qquad \leqslant \mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,B\ldots B} \partial S_n)\, \mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{1,W} \partial S_n) \\
&\qquad \leqslant \mathbb{P}_{1/2}(\partial S_{n_0} \rightsquigarrow_{j,B\ldots B} \partial S_n) \left(\frac{n_0}{n}\right)^{\varepsilon},
\end{aligned}$$

so,

$$\alpha_{j+1} \geqslant \alpha'_j + \varepsilon.$$

Outline

Existence – sketch of the proof (1)

Since the quasi-multiplicativity property holds, it is enough to check that for all $R > 1$

$$\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,BB\ldots B} \partial S_{Rn}) \to f_j(R)$$

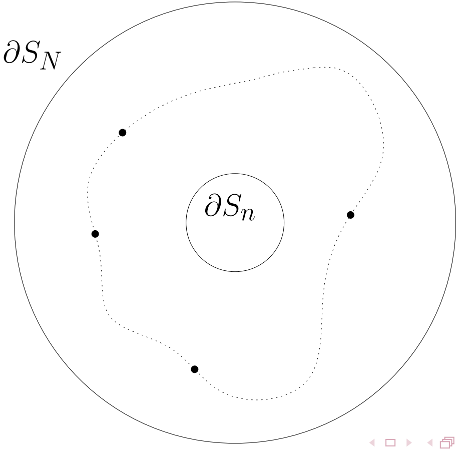
as $n \to \infty$ for some $f_j(R)$.

Indeed, f_j is automatically sub-multiplicative and RSW estimates then show that for any such limit,

$$R^{-1/\varepsilon_j} \leqslant f_j(R) \leqslant R^{-\varepsilon_j}.$$

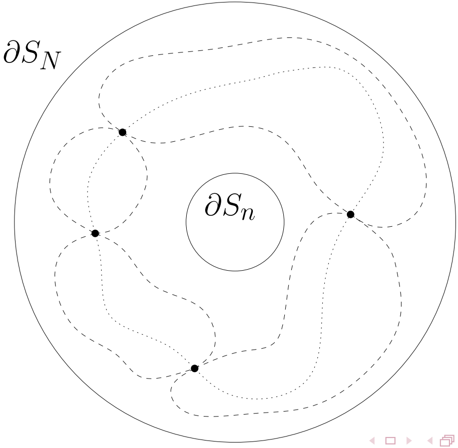
Existence – sketch of the proof (2)

By Menger's theorem, we can write $\{\partial S_n \rightsquigarrow_{j,BB\ldots B} \partial S_{Rn}\}^c$ as the event "There exists a circuit surrounding ∂S_n containing at most $j-1$ black sites".



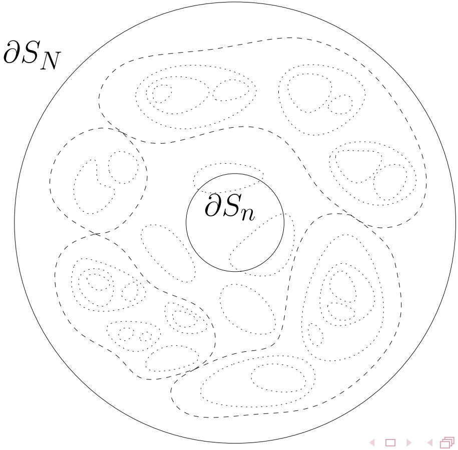
Existence – sketch of the proof (3)

If we look at the cluster interfaces: this event can be expressed as the event "There exists a necklace of at most $j-1$ loops (white inside / black outside) surrounding ∂S_n".



Existence – sketch of the proof (3)

The last formulation goes to the scaling limit, and $f_j(R)$ can be written (implicitly) in terms of the *full scaling limit of percolation* constructed by Camia and Newman:



Outline

Heuristics: energy vs entropy (1)

Basic remark: The expected number of j-arm configurations is the same whatever the sequence of j colors. E.g. for $j = 2$: if

$$N := \#\Big\{ (\gamma_1, \gamma_2) \text{ disjoint s.t. } \gamma_1 \text{ and } \gamma_2 \text{ are black} \Big\}$$

then

$$\mathbb{E}[N] = \sum_{(\gamma_1,\gamma_2)} \left(\frac{1}{2}\right)^{|\gamma_1|} \left(\frac{1}{2}\right)^{|\gamma_2|},$$

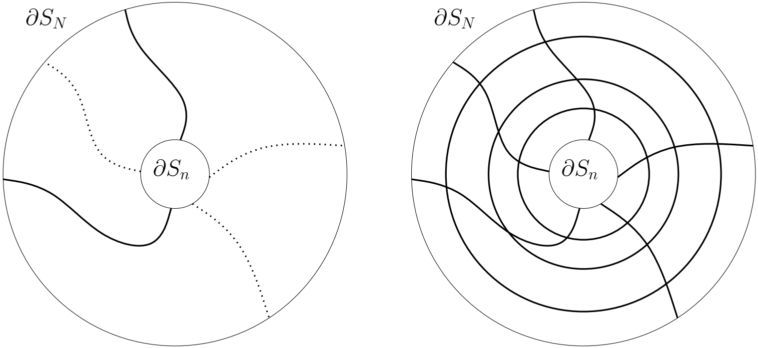
and the same holds for

$$\tilde{N} := \#\Big\{ (\gamma_1, \gamma_2) \text{ disjoint s.t. } \gamma_1 \text{ is black and } \gamma_2 \text{ is white} \Big\}.$$

Heuristics: energy vs entropy (2)

But in a typical configuration with j arms, the number of possible choices for the arms is much larger in the monochromatic case:

- polychromatic case: essentially 1.
- monochromatic case: of order $\log n$.



The set of winding angles

Problem: In both case, many possible *microscopic* modifications (and, not clear at all why there would be as many). How to distinguish two *macroscopically different* choices of arms?

Idea: We define the *set of winding angles* $I_{j,\sigma}$ associated with a realization. For each arm, one can define its winding angle. For any j-arm configuration, we consider the winding angles of the j arms.

$I_{j,\sigma}(n, N)$ is the union over all such j-arm configurations (taken to be empty when there are not j arms with color σ) - enhanced by intervals of length 2π around each point.

The set of winding angles – what we want

By the previous picture, we expect the following behavior:

- polychromatic case: $\mathrm{diam}(I_{j,\sigma}(n,N)) = \mathcal{O}(1)$;
- monochromatic case: $\mathrm{diam}(I_{j,\sigma}(n,N))$ is of order $\log(N/n)$.

Then, we show that this is enough to compare the probabilities that j arms exist in both cases.

Main tool: a correlation inequality (1)

Let $\Omega = \{0,1\}^n$. For $\omega \in \Omega$ and $S \subseteq \{1,\ldots,n\}$, denote

$$[\omega]_S := \{\tilde{\omega} : \forall i \in S,\ \tilde{\omega}_i = \omega_i\}.$$

For any two $A, B \subseteq \Omega$, define the *disjoint occurrence of A and B* as

$$A \circ B := \{\omega : \exists S(\omega) : [\omega]_S \subseteq A \text{ and } [\omega]_{S^c} \subseteq B\}.$$

(We say that S *testifies for A*; definition would be much easier for increasing events !)

Main tool: a correlation inequality (2)

The BKR (Van den Berg, Kesten, Reimer) inequality states that

$$\mathbb{P}(A \circ B) \leqslant \mathbb{P}(A)\mathbb{P}(B).$$

We use an intermediate step in its proof:

Theorem (Reimer)

For any $A, B \subseteq \Omega$, we have

$$|A \circ B| \leqslant |A \cap \bar{B}|,$$

where $\bar{B}$ is the event B with each of the bits "flipped".

Using the correlation inequality

Take $A = \partial S_n \rightsquigarrow_{j-1,B\ldots B} \partial S_N$ and $B = \partial S_n \rightsquigarrow_{1,B} \partial S_N$:

- $A \circ B = \partial S_n \rightsquigarrow_{j,B\ldots B} \partial S_N$,
- $A \cap \bar{B} = \partial S_n \rightsquigarrow_{j,B\ldots W} \partial S_N$.

We thus get $\alpha_j \leqslant \alpha'_j$.

The strict inequality: the setup

By quasi-multiplicativity, it is enough to check that

$$\frac{\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,B\ldots BB} \partial S_N)}{\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,B\ldots BW} \partial S_N)} \to 0$$

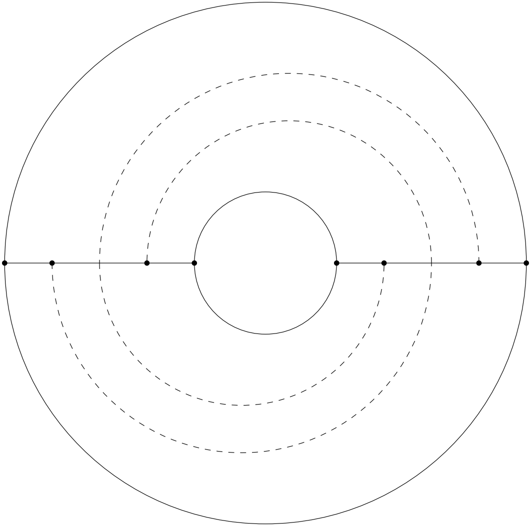
uniformly as $n/N \to 0$. Take $\sigma = B \ldots B$. With high probability,

- $I_{j,\sigma}(n,N) \subseteq [\pm C \log(N/n) \log\log(N/n)]$ (for some $C > 0$);
- $I_{j,\sigma}(n,N)$ contains an interval of length $\varepsilon_j \log(N/n)$ (for some $\varepsilon_j > 0$).

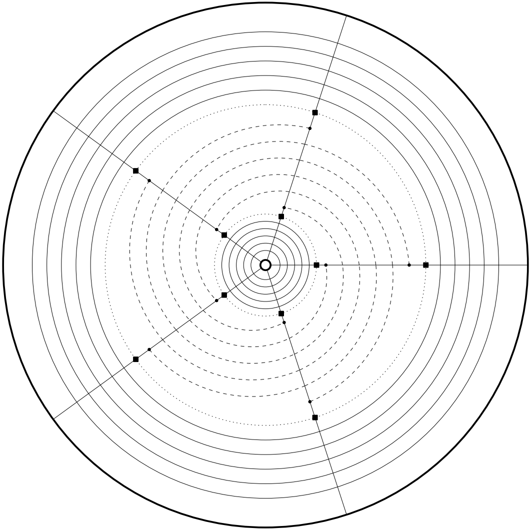
Hence, for some $i_\varepsilon(n,N)$ of width 2π,

$$\mathbb{P}_{1/2}\left(i_\varepsilon(n,N) \subseteq I_{j,\sigma}(n,N) \,\middle|\, \partial S_n \rightsquigarrow_{j,\sigma} \partial S_N\right) \geqslant \frac{C'}{\log\log(N/n)}.$$

Properties of $I_{j,\sigma}(n, N)$ – the case $j = 2$



Properties of $I_{j,\sigma}(n, N)$ – separating the arms



End of the proof (1)

$(6\pi\mathbb{Z}) \cap I_{j,B\ldots BW}(n,N)$ contains at most one point: there exists $\alpha_{\min} \in i_\varepsilon(n,N)$ such that

$$\mathbb{P}_{1/2}(\alpha_{\min} \in I(n,N) \mid \partial S_n \rightsquigarrow_{j,B\ldots BW} \partial S_N) \leqslant \frac{6\pi}{\frac{\varepsilon}{2}\log(N/n)}.$$

Take $A = \{\partial S_n \rightsquigarrow_{j-1,B\ldots B} \partial S_N\} \cap \{\alpha_{\min} \in I_{j-1,B\ldots B}(n,N)\}$ and $B = \partial S_n \rightsquigarrow_{1,B} \partial S_N$:

- $A \circ B = \{\partial S_n \rightsquigarrow_{j,B\ldots B} \partial S_N\} \cap \{\alpha_{\min} \in I_{j,B\ldots BB}(n,N)\}$,
- $A \cap \bar{B} = \{\partial S_n \rightsquigarrow_{j,B\ldots W} \partial S_N\} \cap \{\alpha_{\min} \in I_{j-1,B\ldots B}(n,N)\}$.

End of the proof (2)

$$
\begin{aligned}
&\frac{C'}{\log\log(N/n)}\mathbb{P}(\partial S_n \rightsquigarrow_{j,B\dots BB} \partial S_N) \\
&\leqslant \mathbb{P}_{1/2}(\{\partial S_n \rightsquigarrow_{j,B\dots BB} \partial S_N\} \cap \{\alpha_{\min} \in I_{j,B\dots B}(n,N)\}) \\
&= \mathbb{P}_{1/2}(A \circ B) \\
&\leqslant \mathbb{P}_{1/2}(A \cap \bar{B}) \\
&= \mathbb{P}_{1/2}(\{\partial S_n \rightsquigarrow_{j,B\dots BW} \partial S_N\} \cap \{\alpha_{\min} \in I_{j-1,B\dots B}(n,N)\}) \\
&\leqslant \mathbb{P}_{1/2}(\{\partial S_n \rightsquigarrow_{j,B\dots W} \partial S_N\} \cap \{\alpha_{\min} \in I_{j,B\dots BW}(n,N)\}) \\
&\leqslant \frac{6\pi}{\frac{\varepsilon}{2}\log(N/n)}\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,B\dots BW} \partial S_N).
\end{aligned}
$$

End of the proof (3)

Finally,

$$\frac{\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,B\dots BB} \partial S_N)}{\mathbb{P}_{1/2}(\partial S_n \rightsquigarrow_{j,B\dots BW} \partial S_N)} \leqslant C'' \frac{\log\log(N/n)}{\log(N/n)},$$

which concludes the proof.

A numerical experiment

One can estimate the exponents α'_j by a Monte-Carlo simulation (easily enough for $j \leqslant 3$). The value we obtain are very close to being of the form $k_j/48$ with $k_j \in \mathbb{Z}_+$ (with enormous precision):

$$\alpha'_1 \simeq \frac{5}{48} \; ; \quad \alpha'_2 \simeq \frac{17}{48} \; ; \quad \alpha'_3 \simeq \frac{37}{48}.$$

These values are compatible with the following relation:

$$\alpha'_j = \alpha_j + \alpha'_1 = \alpha_j + \frac{5}{48} = \frac{4j^2+1}{48}.$$

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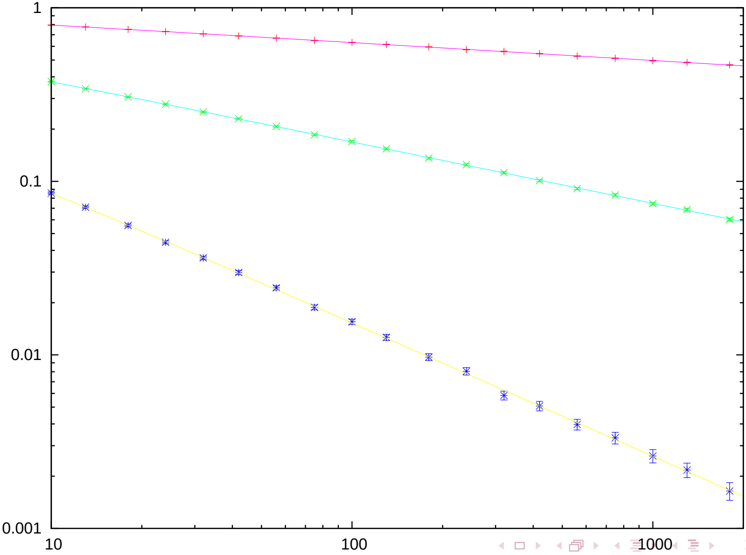
Why?

The results of the simulation



Thank you!